Preface

This reader explains the variables used in traffic modelling, as well as a method to compute delays (vertical queuing using cumulative curves). The third chapter includes a different view of traffic systems, using three different representations of traffic flow theory. These two chapters orgiginate from Knoop (2017). The third chapter is authored by Marie-Jette Wierbos, and included in the reader "Macroscopic Traffic Modelling" (October 2018, TRAIL research school).

Contents

Chapter 1

Variables

This chapter describes the main variables which are used in traffic flow theory. Section 1.1Levels of descriptionsection.1.1 will show the different levels (microscopic, macroscopic and other levels) at which traffic is generally described. Section 1.2Measuring principlessection.1.2 will describe different principles (local, instantaneous and spatio-temporal) to measure the traffic flow. The last section (1.3Stationarity and homogeneitysection.1.3) describes traffic flow characteristics.

1.1 Levels of description

This section will show the different levels at which traffic is generally described. Sections 1.1.1Microscopicsubsection.1.1.1 and 1.1.2Marcoscopicsubsection.1.1.2 will discuss the variables in the microscopic and macroscopic descriptions in more detail.

In a microscopic traffic description, every vehicle-driver combination is described. The smallest element in the description is the vehicle-driver combination. The other often used level of traffic flow description is the macroscopic traffic description. Different from the microscopic description, this level does not consider individual vehicles. Instead, the traffic variables are aggregated over several vehicles or, most commonly, a road stretch. Typical characteristics of the traffic flow on a road stretch are the average speed, vehicle density or flow (see section 1.1.2Marcoscopicsubsection.1.1.2).Other levels of description can also be used, these are described in the last section(see section 1.1.3Other levelssubsection.1.1.3).

1.1.1 Microscopic

In a microscopic traffic description, the vehicle-driver combinations (often referred to as "vehicles", which we will do from now on) are described individually. Full information of a vehicle is given in its trajectory, i.e. the specification of the position of the vehicle at all times. To have full information on these, the positions of all vehicles at all times have to be specified. A graphical representation of vehicle trajectories is given in figure 1.1Vehicle trajectories on a multilane motorwayfigure.caption.2

The trajectories are drawn in a space time plot, with time on the horizontal axis. Note that vehicle trajectories can never go back in time. Trajectories might move back in space if the vehicles are going in the opposite direction, for instance on a two-lane bidirectional rural road. This is not expected on motorways. The slope of the line is the speed of the vehicles. Therefore, the trajectories cannot be vertical – that would mean an infinite speed. Horizontal trajectories are possible at speed zero.

Basic variables in the microscopic representation are speed, headway, and space headway. The speed is the amount of distance a vehicle covers in a unit of time, which is indicated by v. Sometimes, the inverse of speed is a useful measure, the amount of time a vehicle needs for to cover a unit of space; this is called the pace p. Furthermore, there is the space headway or spacing (s) of the vehicle. The net space headway is the distance between the vehicle and its leader. This is also called the gap. The gross space headway of a (following) vehicle the distance *including* the

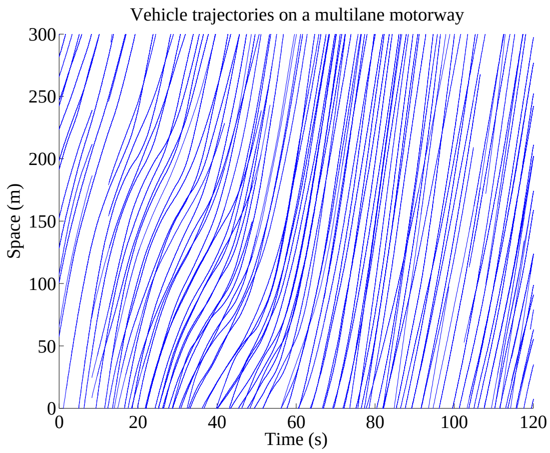


Figure 1.1: Vehicle trajectories on a multilane motorway

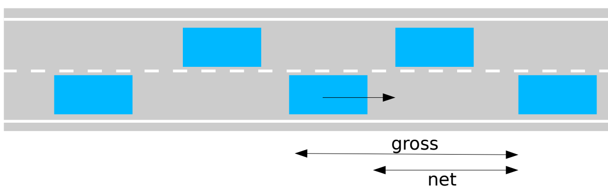


Figure 1.2: The difference between gross and net spacing (or headway)

length of the vehicle, so the distance from the rear bumper of the leading vehicle to the rear bumper of the following vehicle. Similarly, we can time it takes for a follower to get to reach (with its front bumper) the position of its leader's rear bumper. This is called the net time headway. If we also add the time it cost to cover the distance of a vehicle length, we get the gross time headway. See also figure 1.2The difference between gross and net spacing (or headway)figure.caption.3. The symbol used to indicate the headway is h.

From now on, in this reader we will use the following conventions:

- Unless specified otherwise, headway means time headway
- Unless specified otherwise, headways and spacing are given as gross values

Figure 1.3The microscopic variables explained based on two vehiclesfigure.caption.4 shows the variables graphically. The figure shows two vehicles, a longer vehicle and a shorter vehicle. Note that the length of the vehicles remains unchanged, so the difference between the gross and net spacing is the same, namely the vehicle length. However, the difference between the gross and net time headway changes based on the vehicle speed.

In a trajectory plot, the slope of the line is the speed. If this slope changes, the vehicle accelerates or decelerates. So, the curvature of the lines in a trajectory plot shows the acceleration or deceleration of the vehicle. If the slope increases, the vehicle accelerates, if it decreases, it decelerates.

1.1.2 Marcoscopic

In a macroscopic traffic description, one does not describe individual vehicles. Rather, one describes for each road section the aggregated variables. That is, one can specify the density k, i.e. how close in space vehicles are together. Furthermore, one can specify the flow q i.e. The number of vehicles passing a reference point per unit of time. Finally, one can describe the average speed u of the vehicles on a road section. Other words for flow are thoughput, volume or intensity; we will strictly adhere to the term flow to indicate this concept.

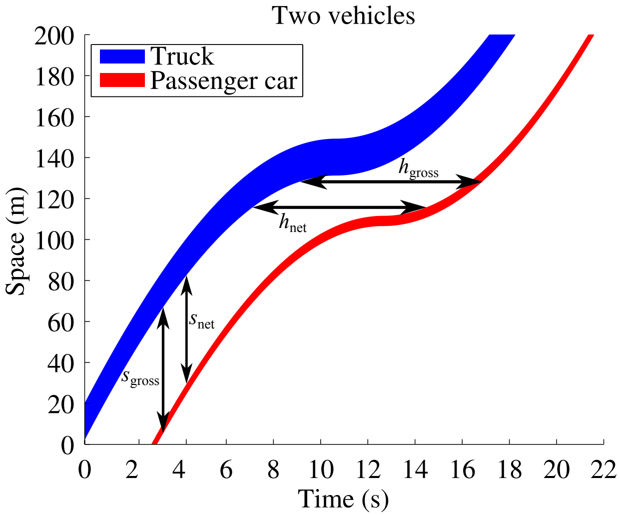


Figure 1.3: The microscopic variables explained based on two vehicles

Table 1.1: Overview of the microscopic and macroscopic variables and their relationship; the pointy brackets indicate the mean.

Microscopic	symbol	unit	Macroscopic	symbol	unit	relation
Headway	h	s	Flow	q	vtg/h	$q = \dfrac{3600}{\langle h \rangle}$
Spacing	s	m	Density	k	vtg/km	$k = \dfrac{1000}{\langle s \rangle}$
Speed	v	m/s	Average speed	u	km/u	$u = 3.6 \langle v \rangle$

All of the mentioned macroscopic variables have their microscopic counterpart. This is summarized in table 1.1Overview of the microscopic and macroscopic variables and their relationship; the pointy brackets indicate the mean.table.caption.5. The density is calculated as one divided by the average spacing, and is calculated over a certain road stretch. For instance, if vehicles have a spacing of 100 meters, there are 1/100 vehicles per meter, or 1000/100=10 veh/km. The flow is the number of vehicles that pass a point per unit of time. It can be directly calculated from the headways by dividing one over the average headway. For instance, if all vehicles have a headway of 4 seconds, there are 1/4 vehicles per second. That means there are 3600(s/h)/4(s/veh)= 900 veh/h. In table 1.1Overview of the microscopic and macroscopic variables and their relationship; the pointy brackets indicate the mean.table.caption.5 units are provided and in the conversion from one quantity to the other, one needs to pay attention. Note that provided units are not obligatory: one can present individual speed in km/h, or density in veh/hm. However, always pay attention to the units before converting or calculating.

Relation to the microscopic level

The average speed is calculated as an average of the speeds of vehicles at a certain road stretch. This speed differs from the average speed obtained by averaging speed of all vehicles passing a certain point. The next section explains the different measuring principles. The full explanation of the differences between the two speeds and how one can approximate the (space) average speed by speeds of vehicles passing a certain location is presented in section **??**.

Another concept for a traffic flow, in particular in relation to a detector (see also section 1.2Measuring principlessection.1.2), is the occupancy o. This indicates which fraction of a time a detector embedded in the roadway is occupied, i.e. whether there is a vehicle on top of the detector. Suppose a detector has a length L_{det} and a vehicle a length of L_i. The occupancy is defined as the time the detector is occupied, τ_{occupied} divided by all time, i.e. the time it is occupied and

time is is not occupied $\tau_{\text{not occupied}}$

$$o = \frac{\tau_{\text{occupied}}}{\tau_{\text{occupied}} + \tau_{\text{not occupied}}} \tag{1.1}$$

The occupation time can be derived from the distances and the speed. The distance the vehicle has to cover from the moment it starts occupying the detector up to the time it leaves the detector is its own length plus the length of the detector. Hence, the occupancy time is

$$\tau_{\text{occupied}} = \frac{L_i + L_{\text{det}}}{v} \tag{1.2}$$

Once the first vehicle drives off the detector, the distance for the following vehicle to reach the detector is the gap (i.e., the spacing minus the length of the vehicle) between the vehicles minus the length of the detector. The amount of time this takes is

$$\tau_{\text{not occupied}} = \frac{s - L_i - L_{\text{det}}}{v} \tag{1.3}$$

Substituting the expressions for the occupancy time and the non-occupancy time into equation 1.1Relation to the microscopic levelequation.1.1.1 and rearranging the terms, we get

$$o = \frac{L_i + L_{\text{det}}}{s} \tag{1.4}$$

In practice, the detector length is known for a certain road configuration (usually, there are country specific standards). So assuming a vehicle length, one can calculate the spacing, and hence the density, from the occupancy.

1.1.3 Other levels

Apart from the macroscopic and microscopic traffic descriptions, there are three other levels to describe traffic. They are are less common, and are therefore not discussed in detail. The levels mentioned here are mainly used in computer simulation models.

Mesoscopic

The term mesoscopic is used for any description of traffic flow which is in-between macroscopic and microscopic. It can also be a term for simulation models which calculate some elements macroscopically and some microscopically. For instance, dynasmart Dynasmart (2003), uses such a mesoscopic description.

Submicroscopic

In a submicroscopic description the total system state is determined by the sub levels of a vehicle and/or driver. Processes which influence the speed of a vehicle, like for instance mechanically throttle position and engine response, or psychologically speed perception, are explicitly modelled. This allows to explicitly model the (change in) reaction on inputs. For instance, what influence would cars with a stronger engine have on the traffic flow.

Network level

A relatively new way of describing the traffic state is the network level. This has recently gained attention after the publication by Geroliminis and Daganzo (2008). Instead of describing a part of a road as smallest element, one can take an area (e.g. a city center) and consider this as one unit. Chapter **??** is devoted to this description.

1.2 Measuring principles

Whereas the previous sections described which variables are used to describe traffic flow, this section will introduce three principles of measuring the traffic flow. These principles are local, instantaneous and spatio-temporal.

1.2.1 Local

With local measurements one observes traffic at one location. This can be for instance a position at the roadway. To measure motorway traffic, often inductive loops are used. These are coils embedded in the pavement in which a electrical current produces a (vertical) magnetic field. If a car enters or leaves this magnetic field, this can be measured in the current of the coil. Thus, one knows how long a loop is occupied. In the US, usually single loops are used, giving the occupancy of the loop. Using equation 1.4Relation to the microscopic levelequation.1.1.4, this can be translated into density. The detectors also measure the flow. As will be explained later in section **??**, this suffices to completely characterise the traffic flow.

This determination of densitybuilds upon the assumption of the vehicle length being known. One can also measure the length of a vehicle for passing vehicles, using dual loop detectors. These are inductive loops which are placed a known short distance (order of 1 m) from each other. If one measures carefully the time between the moment the vehicle starts occupying the first loop and the moment it starts occupying the second loop, one can measure its speed. If its speed is known, as well as the time it occupies one loop, the length of the vehicle can also be determined.

1.2.2 Instantaneous

Contrary to local measurements, there are instantaneous measurements. These are measurements which are taken at one moment in time, most likely over a certain road stretch. An example of such a measurement is an areal photograph. In such a measurement, one can clearly distinguish spatial characteristics, as for instance the density. However, measuring the temporal component (flow) is not possible.

1.2.3 spatio-temporal measurements

Apart from local or instantaneous measurements, one can use measurements which stretch over a period of time *and* a stretch of road. For instance, the trajectories in figure 1.1Vehicle trajectories on a multilane motorwayfigure.caption.2 are an example thereof. This section will introduce Edie's definitions of flow, density and speed for an area in space and time.

A combination of instantaneous measurements and local measurements can be found in remote sensing observations. These are observations which stretch in both space and time. For instance, the trajectories presented in figure 1.1Vehicle trajectories on a multilane motorwayfigure.caption.2 can be observed using a camera mounted on a high point or a helicopter. One can see a road stretch, and observe it for a period of time.

Measuring average speed by definition requires an observation which stretches over time and space. At one location, one cannot determine speed, nor at one moment. One needs at least two locations close by (several meters) or two time instances close by. Ignoring these short distances one can calculate a local mean speed based on speeds of the vehicles passing by location. Ignoring the short times, one can calculate the time mean speed from the speed of the vehicles currently at the road. At this moment, we suffice by mentioning these average speeds are different. Section **??** will show how the space mean speed can be approximated from local measurements.

Figure 1.4Vehicle trajectories and the selection of an area in space and timefigure.caption.6 shows the same trajectories as figure 1.1Vehicle trajectories on a multilane motorwayfigure.caption.2, but in figure1.4Vehicle trajectories and the selection of an area in space and timefigure.caption.6 an area is selected. Trajectories within this area in space and time are coloured red. Note that an selected area is not necessarily square. It is even possible to have a convex area, or boundaries moving backwards and forwards in time. The definitions as introduced here will hold for all types of areas, regardless of their shape in space-time.

Let us consider the area X. We indicate its size by W_X, which is expressed in km-h, or any other unit of space times time. For all vehicles, we consider the distance they drive in area X, which we call $d_{X,i}$. Adding these for all vehicles i gives the total distance covered in area X, indicated by TD:

$$TD = \sum_{\text{all vehicles } i} d_{X,i} \tag{1.5}$$

For a rectangular area in space and time, the distance covered might be the distance from the upstream end to the downstream end, but the trajectory can also begin and/or end at the side of the area, at a certain time. In that case, the distance is less than the full distance.

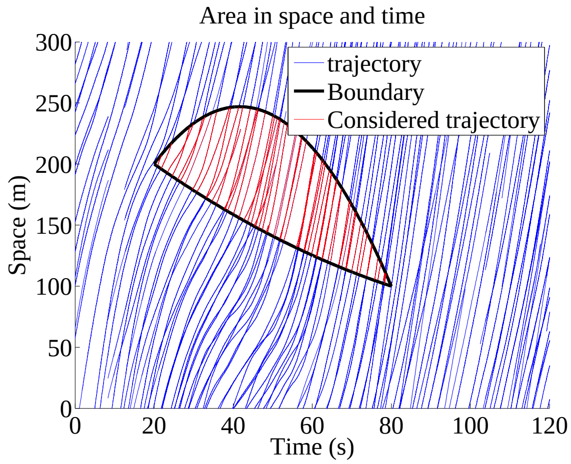


Figure 1.4: Vehicle trajectories and the selection of an area in space and time

Similarly, we can define the time a vehicle spends in area X, $t_{X,i}$, which we can sum for all vehicles i to get the total time spent in area X, indicated by TT.

$$TT = \sum_{\text{all vehicles } i} t_{X,i} \tag{1.6}$$

Obviously, both quantities grow in principle with the area size. Therefore, the traffic flow is best characterised by the quantities TD/W_X and TT/W_X. This gives the flow and the density respectively:

$$q = \frac{TD}{W_X} \tag{1.7}$$

$$k = \frac{TT}{W_X} \tag{1.8}$$

Intuitively, the relationship is best understood reasoning from the known relations of density and flow. Starting with a situation of 1000 veh/h at a cross section, and an area of 1 h and 2 km. In 1 hour, 1000 vehicles pass by, which all travel 2 kilometres in the area (There the vehicles which cannot cover the 2 km because the time runs out, but there are just as many which are in the section when the time window starts). So the total distance is the flow times the size of the area: $TD = qW_X$. This can be simply rewritten to equation 1.8spatio-temporal measurementsequation.1.2.7.

A similar relation is constructed for the density, considering again the rectangular area of 1 hour times 2 kilometres. Starting with a density of 10 veh/km, there are 20 vehicles in the area, which we all follow for one hour. The total time spent, is hence 10*2*1, or $TT = kW_X$. This can be rewritten to equation 1.8spatio-temporal measurementsequation.1.2.7.

The average speed is defined as the total distance divided by the total time, so

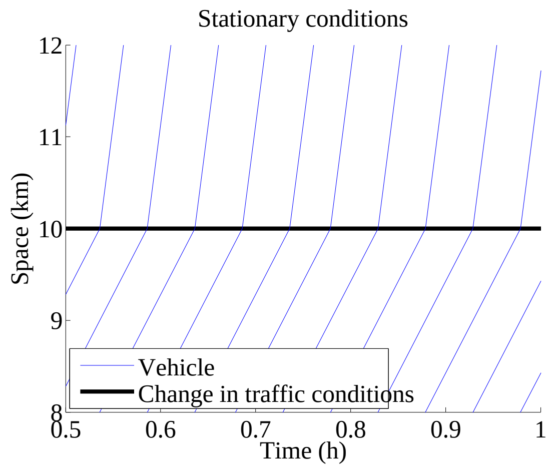
$$u = \frac{TD}{TT} \tag{1.9}$$

The average travel time over a distance l can be found as the average of the time a vehicle travels over a distance l. In an equation, we find:

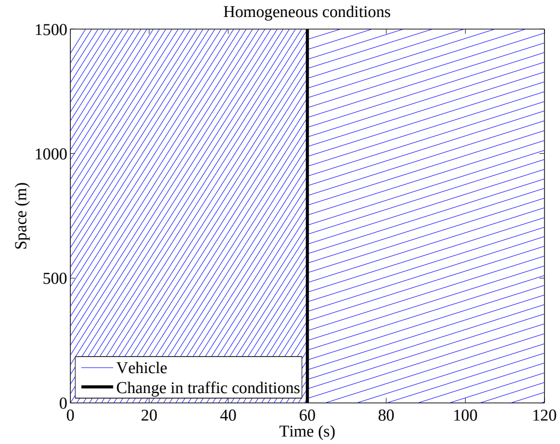
$$\langle tt \rangle = \left\langle \frac{l}{v} \right\rangle = l \left\langle \frac{1}{v} \right\rangle \tag{1.10}$$

In this equation, tt indicates the travel time and the pointy brackets indicate the mean. This can be measured for all vehicles passing a road stretch, for instance at a local detector. Note that the mean travel time is *not* equal to the distance divided by the mean speed:

$$\langle tt \rangle = l \left\langle \frac{1}{v} \right\rangle \neq l \frac{1}{\langle v \rangle} \tag{1.11}$$



(a) Stationary conditions



(b) Homogeneous conditions

In fact, it can be proven that in case speeds of vehicles are not the same, the average travel time is underestimated if the mean speed is used.

$$\langle tt \rangle = l \left\langle \frac{1}{v} \right\rangle \leq l \frac{1}{\langle v \rangle} \tag{1.12}$$

The harmonically averaged speed (i.e., 1 divide by the average of 1 divided by the speed) does provide a good basis for the travel time estimation. In an equation, we best first define the pace, p_i:

$$p_i = \frac{1}{v_i} \tag{1.13}$$

The harmonically averaged speed now is

$$\langle v \rangle_{\text{harmonically}} = \frac{1}{\langle p \rangle} = \frac{1}{\left\langle \frac{1}{v_i} \right\rangle} \tag{1.14}$$

The same quantity is required to find the space mean speed. Section **??** shows the difference qualitatively. In short, differences can be several tens of percents.

1.3 Stationarity and homogeneity

Traffic characteristics can vary over time and/or over space. There are dedicated names for traffic if the state does not change.

Traffic is called stationary if the traffic flow does not change over time (but it can change over space). An example can be for instance two different road sections with different characteristics. An example is given in figure 1.5(a)Subfigure 1 1.5(a)subfigure.1.5.1, where there first is a low speed, then the speed of the vehicles is high.

Traffic is called homogeneous if the traffic flow does not change over space (but it can change over time). An example is given in figure 1.5(b)Subfigure 1 1.5(b)subfigure.1.5.2,where at time 60 the speed decreases at the whole road section. This is much less common than the stationary conditions. For this conditions to occur, externally the traffic regulations have to change. For instance, the speed limits might change at a certain moment in time (lower speeds at night).

Chapter 2

Cumulative curves

This chapter discusses cumulative curves, also known as cumulative flow curves. The chapter first defines the cumulative curves (section 2.1), then it is show how traffic characteristics can be derived from these (section 2.2). Section 2.4 shows the application of slanted cumulative curves.

2.1 Defenition

The function$N_x(t)$ is defined as the number of vehicles that have passed a point x at time t and is only used for traffic into one direction. Hence, this function only increases over time. Strictly speaking, this function is a step function increasing by one every time a vehicle passes. However, for larger time spans and higher flow rates, the function is often smoothed into a continuous differentiable function.

The increase rate of this function equals the flow:

$$\frac{dN}{dt} = q \tag{2.1}$$

Hence from the flow, we can construct the cumulative curve:

$$N = \int q dt \tag{2.2}$$

This gives one degree of freedom, the value to start at. This can be chosen freely, or should be adapted to cumulative curves for other locations.

2.2 Vertical queuing model

A vertical queuing model is a model which assumes an unlimited inflow and and an outflow which is restricted to capacity. The vehicles which cannot pass the bottleneck are stacked "vertically" and do not occupy any space. Figure 2.1 illustrates this principle.

Let us now study the dynamics of such a queue. We discretize time in steps of duration Δt, referred to by index t. The demand is externally given, and indicated by D. At time steps t we compute the flow into and out of the stack (the number of vehicles in the stack indicated as S). In between the time steps, indicated here as $t + 1/2$, the number of vehicles in the stack is updated based on the flows q. Then, the stack provide the basis for the flows in the next time step.

The stack starts at zero. Then, for each time step first the inflow to the stack is computed.

$$q_{\text{in},t} = D \tag{2.3}$$

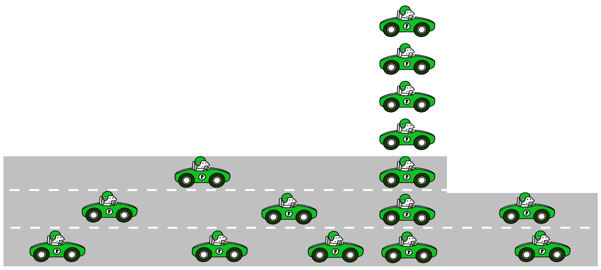
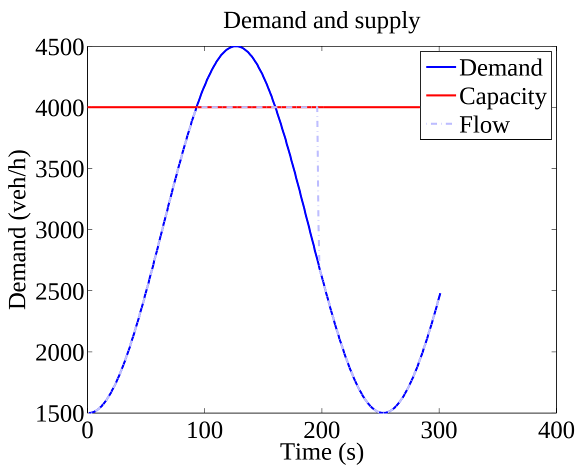
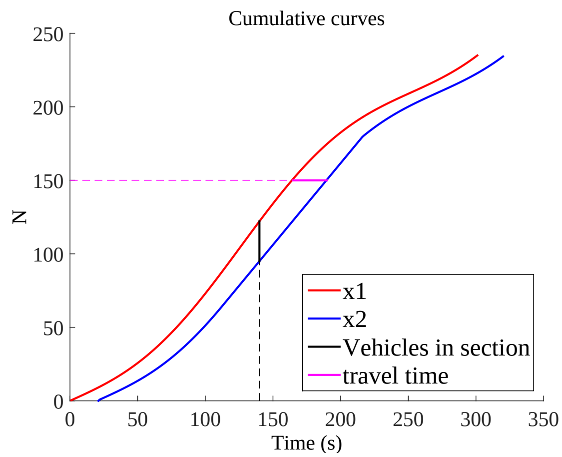


Figure 2.1: Illustration of a vertical queue



(a) Demand and supply



(b) Cumulative curves

Figure 2.2: Demand and cumulative curves

and the stack is updated accordingly, going to an intermediate state at time step t+1/2. This intermediate step is the number of vehicles in the queue if there were no outflow, so the original queue plus the inflow:

$$S_{t+1/2} = S_t + q_{\text{in}}\Delta t \tag{2.4}$$

Then, the outflow out of the stack (q_{out}) is the minimum of the number of vehicles in this intermediate queue and the maximum outflow determined by the capacity C:

$$q_{\text{out}} = \min\{C\Delta t, S_{i+1/2}\} \tag{2.5}$$

The stack after the time step is then computed as follows

$$Si+1 = S_{i+1/2} - q_{\text{out}}\Delta t = S_i + (q_{\text{in},i} - q_{\text{out},i})\,\Delta t \tag{2.6}$$

Let us consider a situation as depicted in figure 2.1, and we are interested in the delays due to the bottleneck with a constant capacity of 4000 veh/h. The demand curve is plotted in figure 2.2(a). The flows are determined using the vertical queuing model . The flows are also show in figure 2.2(a). Note that the area between the flow and demand curve where the demand is higher than the flow (between approximately 90 to 160 seconds), is the same as the area between the curves where the flow is higher than the demand (between approximately 160 and 200 seconds). The reasoning is that the area represents a number of vehicles (a flow times a time). From 90 to 160 seconds the demand is higher than the flow, i.e., the inflow is higher than the outflow. The area represents the number of vehicles that cannot pass the bottleneck, and hence the number of queued vehicles. From 160 seconds, the outflow of the queue is larger than the inflow. That area represents the number of vehicles that has left the queue, and cannot be larger than the number of vehicles queued. Moreover, the flows remains at capacity until the stack is empty, so both areas must be equal.

2.3 Travel times, densities and delays

This section explains how travel times and delaysDelay can be computed using cumulative curves. Note that this methodology does not take spillback effects into account. If one requires this to be accounted for, please refer to shockwave theory (chapter **??**).

2.3.1 Construction of cumulative curves

The cumulative curves for the above situation is shown in figure 2.2(b). The curves show the flows as determined by the vertical queuing model. The for the inflow we hence use equation 2.3 and for the outflow we use 2.5; for both, the cumulative curves are constructed using equation 2.2.

2.3.2 Travel times, number of vehicles in the section

A black line is drawn at $t = 140s$ in figurefigure 2.2(b) . The figure shows by intersection of this line with the graphs how many vehicles have passed the upstream point $x1$ and how many vehicles have passed the downstream point $x2$. Consequently, it can be determined how many vehicles are in the section between $x1$ and $x2$. This number can also be found in the graph, by taking the difference between the inflow and the outflow at that moment. This is indicated in the graph by the bold vertical black line.

Similarly, we can take a horizontal line; consider for instance the line at $N = 150$. The intersection with the inflow line shows when the 150th vehicle enters the section, and the intersection with the outflow line shows when this vehicle leaves the section. So, the horizontal distance between the two lines is the travel time of the 150th vehicle. At times where the demand is lower than the capacity, the vehicles have a free flow travel time. So without congestion, the outflow curve is the inflow curve which is translated to the right by the free flow travel time.

The vertical distance is the number of vehicles in the section (ΔN)at a moment t. In a time period dt this adds $\Delta N dt$ to the total travel time (each vehicle contributes dt). To get the total travel time, we integrate over all infinitesimal intervals dt:

$$tt = \int \Delta N dt \tag{2.7}$$

The horizontal distance between the two lines is the travel time for one vehicle, and vertically we find the number of vehicles. Adding up the travel times for all vehicles gives the total travel time:

$$tt = \sum_i tt_i \tag{2.8}$$

In a continuous approach, this changes into

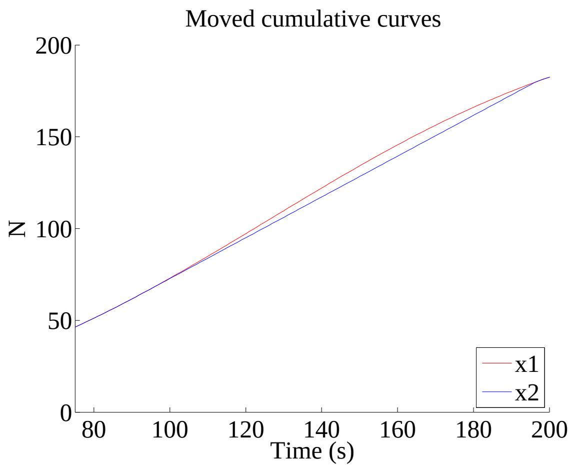
$$tt = \int tt_i di \tag{2.9}$$

Both calculation methods lead to the same interpretation: *the total time spent can be determined by the area between the inflow and outflow curve*.

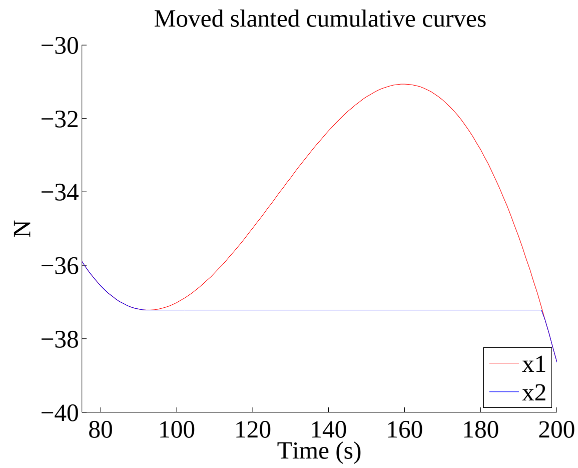
2.3.3 Delays

Delays for a vehicle are the extra time it needs compared to the free flow travel time. so to calculate delay, one subtracts the free flow travel time from the actual travel time. To subtract the free flow travel time from the travel time, we can graphically move the outflow curve to the left, as is shown in figure 2.3(a). For illustration purposes, the figure is zoomed at figure 2.3(b). The figure shows that if the travel time equals the free flow travel time, both curves are the same, leading to 0 delay.

Similar to how the cumulative curves can be used to determine the travel time, the moved cumulative curves can be used to determine the delay. The delay for an individual vehicle can be found by the horizontal distance between



(a) Moved cumulative curves



(b) Slanted cumulative curves

Figure 2.3: Determining the delay and the flows from cumulative curves

the two lines. The vertical distance between the two lines can be interpreted as the number of vehicles queuing. The total delay is the area between the two lines:

$$\mathcal{D} = \int tt_i - tt_{\text{free flow}} di \tag{2.10}$$

This is the area between the two lines. If we define N_{queue} as the number of vehicles in the queue at moment t, we can also rewrite the total delay as

$$\mathcal{D} = \int N_{queue}(t)dt \tag{2.11}$$

2.4 Slanted cumulative curves

Slanted cumulative curves or oblique cumulative curves is a very powerful yet simple tool to analyse traffic streams. These are cumulative curves which are off set by a constant flow:

$$\tilde{N} = \int q - q_0 dt - \int q_0 dt = \int q dt - \int q_0 dt \tag{2.12}$$

This means that differences with the freely chosen reference flow q_0 are amplified: in fact, only the difference with the reference flow are counted. The best choice for the reference flow q_0 is a capacity flow.

Figure 2.3(b) shows the slanted cumulative curves for the same situation as in figure 2.3(a). The figure is off set by $q_0 = 4000$ veh/h. Because the demand is initially lower than the capacity, $\tilde{N}$ reaches a negative value. From the moment outflow equals capacity, the slanted cumulative outflow curve is constant. Since the demand is higher than the capacity, this increases. At the moment both curves intersect again, the queue is dissolved.

The vertical distance between the two lines still shows the length of the queue, N_{queue}. That means that equation 2.10 still can be applied in the same way for the slanted cumulative curves, and the delay is the area between the two lines.

Slanted cumulative curves are also particularly useful to determine capacity, and to study changes of capacity, for instance the capacity drop (see section **??**). In that case, for one detector the slanted cumulative curves are drawn. By a change of the slope of the line a change of capacity is detected. In appendix A a Matlab code is provided by which cumulative curves can be made, and which includes the computation of several key performance indices.

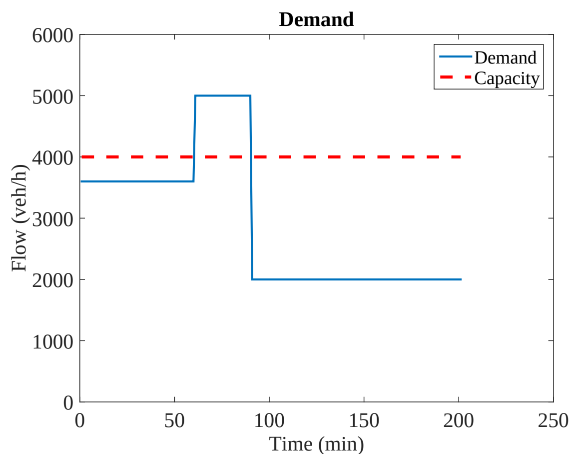


Figure 2.4: Demand and capacity

2.5 Practical limitations

Cumulative curves are very useful for models where the blocking of traffic does not play a role. For calculating the delay in practise, the method is not very suitable due to failing detectors. Any error in the detection (a missed or double counted observation), will change one of the curves and will offset the cumulative flow, and this is never corrected; this is called cumulative drift. Recently, an algorithm has been proposed to check the offsets by cross checking the cumulative curves with observed travel times (Van Lint et al., 2014). This is work under development. Moreover, some types of detectors will systematically miscount vehicles, which makes the above-mentioned error larger.

Apart from their use in models, slanted cumulative curves are very powerful to show changes in capacity in practise.

2.6 Example application

Consider a road with a demand of:

$$q_{\text{in}} = \begin{cases} 3600v/h & \text{for } t < 1h \\ 5000v/h & \text{for } 1h < t < 1.5h \\ 2000v/h & \text{for } t > 1.5h \end{cases} \tag{2.13}$$

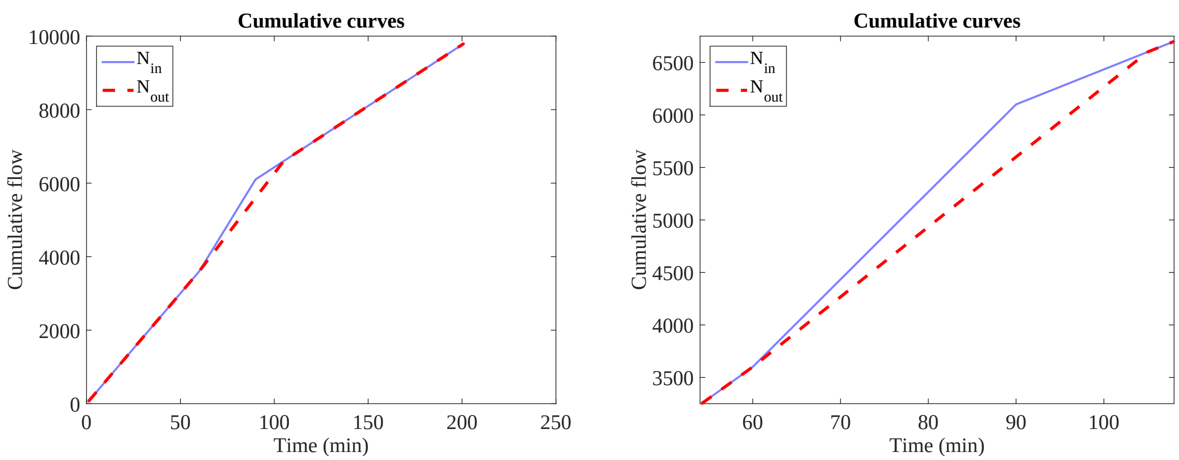
The capacity of the road is 4000 veh/h. A graph of the demand and capacity is shown in figure 2.4.

1. Construct the (translated=moved) cumulative curves
2. Calculate the first vehicle which encounters delay (N)
3. Calculate the time at which the delay is largest
4. Calculate the maximum number of vehicles in the queue
5. Calculate the vehicle number (N) with the largest delay
6. Calculate the delay this vehicle encounters (in h, or mins)
7. Calculate the time the queue is solved
8. Calculate the last vehicle (N) which encounters delay

9. Calculate the total delay (veh-h)
10. Calculate the average delay of the vehicles which are delayed (h)

This can be answered by the following:

1. For the cumulative curves, an inflow and an outflow curve needs to be constructed; both increase. For the inflow curve, the slope is equal to the demand. For the outflow curve, the slope is restricted to the capacity. During the first hour, the demand is lower than the capacity, hence *the outflow is equal to the demand*. From t=1h, the inflow exceeds the capacity and the outflow will be equal to the demand. The cumulative curve hence increases with a slope equal to the capacity. As long as there remains a queue, i.e. the cumulative inflow is higher than the outflow, the outflow remains at capacity. The outflow remains hence increasing with a slope equal to the capacity until it intersects with the cumulative inflow. Then, the outflow follows the inflow: see figure 2.5(a) and for a more detailed figure 2.5(b).
2. The first vehicle which encounters delay (N) Delays as soon as q¿C: so after 1h at 3600 v/h = 3600 vehicles.
3. : The time at which the delay is largest: A queue builds up as long as q¿C, so up to 1.5 h. At that moment, the delay is largest
4. The maximum number of vehicles in the queue: 0.5 h after the start of the queue, 0.5*5000=2500 veh entered the queue, and 0.5*4000=2000 left: so 500 vehicles are in the queue at t=0.5h
5. The vehicle number (N) with the largest delay: N(1.5h)=3600+0.5*5000 = 6100
6. : The delay this vehicle encounters (in h, or mins): It is the 2500th vehicle after t=1h. The delay is the horizontal delay between the entry and exit curve. It takes at capacity 2500/4000 = 37.5 mins to serve 2500 vehicles. It entered 0.5 hours = 30 mins after t=1, so the delay is 7.5 mins
7. The time the queue is solved: This is the time point that the inflow and outflow curves intersect again. 500 vehicles is the maximum queue length, and it reduces with 4000-2000=2000veh/h. So 500/2000=15 minutes after the time that q¡C the queue is solved, i.e. 1:45h after the start.
8. The last vehicle (N) which encounters delay This is the verhicle number at the moment the inflow and outflow curves meet again. 15 minutes after the vehicle number with the largest delay: 6100+0.25*2000 = 6600 veh
9. The total delay. This is the area of the triangle between inflow and outflow curve. This area is computed by 0.5 * height * base = 0.5 * 500 * (30+15)/60 = 187,5 veh-h. Note that here we use a generalised equation for the area of a triangle. Indeed, we transform the triangle to a triangle with a base that has the same width, and the hight which is the same for all times (i.e., we skew it). The hight of this triangle is 500 vehicles (the largest distance between the lines) and the width is 45 minutes.
10. The average delay of the vehicles which are delayed (h) 187,5 veh-h/ (6600-3600) veh = 0,0625 h = 3,75 min



(a) Overview of the cumulative curves

(b) More detailed view of the cumulative curves

Figure 2.5: Cumulative curves for the example

Chapter 3

Traffic state dynamics in three representations

Maria J. Wierbos and Victor L. Knoop

3.1 Different representations

Traffic can be described by a fixed relation between X, N and T. The most common way to describe traffic is the N-model using Eulerian coordinates, which describes the number of vehicles (N) that have passed location x at time t. Another well-known representation is the X-model in Lagrangian coordinates, which describes the position (X) of vehicle n at time t. The third and least common representation is the T-model, which describes the time (T) at which vehicle n crosses location x (Laval and Leclercq (2013)). All three models describe the same traffic state, for example the situation shown in figure 3.1. The example displays the journey of around 75 vehicles on a single lane road. The 5th vehicle stops for around 60 timesteps and creates a jam, which slowly dissolves. With use of this example, the describing parameters, characteristics and shockwave theory in the 3 different approaches are described in the following sections.

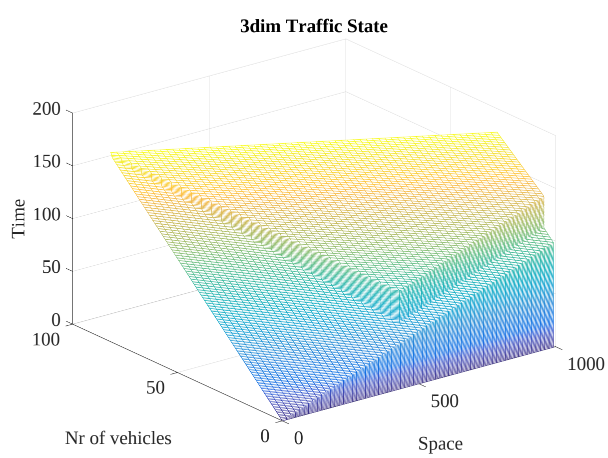


Figure 3.1: Example of a traffic situation, expressed in x, n and t

3.1.1 Describing parameters

In the different models, the traffic state are be explained in different combinations of x, n and t. The derivatives of these parameters give a first insight to the important variables. For example, the change in vehicle number n with time t is the flow q and the change of vehicle number n with space x is the density k. An overview of the other variables are presented in table 3.2. The pitfall in determining the derivatives are the correct signs, which originate from the convention of the scales. The convention of space can be either positive or negative, but time is always positive. For vehicle number, the convention is that the first vehicle on a road has the lowest n number. As a result, the higher vehicle numbers correspond to lower x values, and the derivative dN/dx has a minus sign.

Table 3.1: Variables used in different coordinate systems

	$N(x,t)$	$X(n,t)$	$T(n,x)$
$/dt$	$q(x,t)$	$v(n,t)$	
$/dx$	$-k(x,t)$		$p(n,x)$
$/dn$		$-s(n,t)$	$h(n,x)$

3.2 Model characteristics

In this section the characteristics of the three models are described in more detail. The shape of the fundamental diagram is presented first, followed by the interpretation of trajectories, conservation equations and shockwave theory. For this sections, the based knowledge of traffic flow theory is assumed. Please look at the course reader of '*CIE 4821 Traffic Flow Theory and Simulation*' for further information.

3.2.1 N-model

The most common way to describe traffic is the N-model. In this model the flow q is proportional to the density k and speed u, with $q = ku$. A fundamental diagram can be drawn for flow where density is the main variable to determine the flow. Different shapes of the fundamental diagram are proposed to best capture the traffic behavior but for simplicity reasons only the triangular fundamental diagram is discussed here. The characteristics of the triangular fundamental diagram (Figure 3.2a) for flow as function of density $Q(k)$ are:

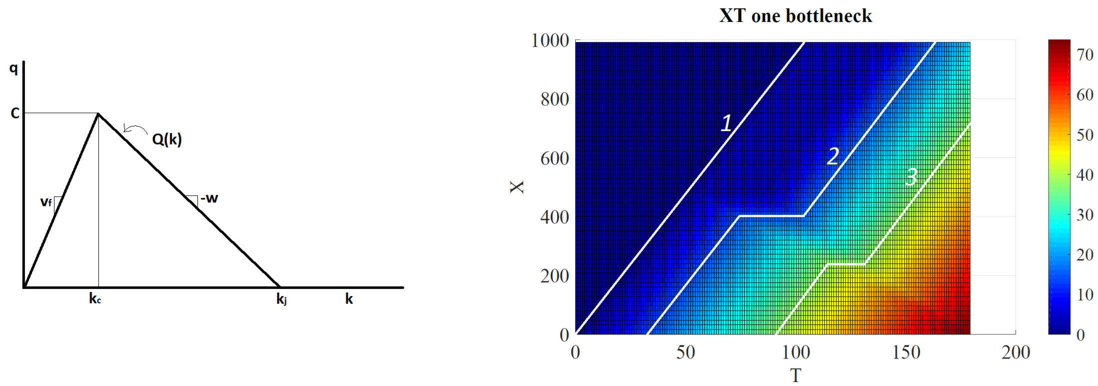
- Zero flow at jam density and zero density.
- A maximum flow value C when the density is at $k = k_c$
- A free flow branch for $0 < k \leq k_c$. In this range, flow increases linearly with density until it reaches the critical density k_c. The increase is equal to the free flow speed v_f.
- A congestion branch when density exceeds the critical density $k > k_c$. In this range, flow decreases with wave speed $-w$ until the density reaches jam density k_j

Trajectories in the N-model are iso-n lines which represent the movement of an individual vehicle in time and space. The XT-diagram in Figure 3.2b shows model output of around 75 vehicles driving on a 1 lane road, which is the same situation as in Figure 3.1. The 5th vehicle stops at $x = 450$ for around 60 timesteps, causing a jam. The trajectories of 3 cars are isolated, the first having an undisturbed path, the second trajectory is stationary for around 40 timesteps before continuing its path, and the third trajectory is stationary at a smaller x-location for a shorter time period. This indicates that the jam grows in the $-x$ direction, and that the queue length dissolves with time. In this model data, but also in observed trajectory data, the vertical distance between two trajectories provides information about the density, using $\frac{\Delta x}{\Delta N} = -\frac{1}{k}$. The horizontal distance between trajectories is a measure for the flow $\frac{\Delta t}{\Delta N} = \frac{1}{q}$.

Assuming that all vehicles that enter a certain road stretch also have to exit the road stretch within a certain amount of time, it can be stated that the number of vehicles are conserved. The conservation equation is given by:

$$\frac{\delta k}{\delta t} + \frac{\delta q}{\delta x} = 0$$

which states that the change in density k with time t and the change of flow q with distance x should equal out to zero.



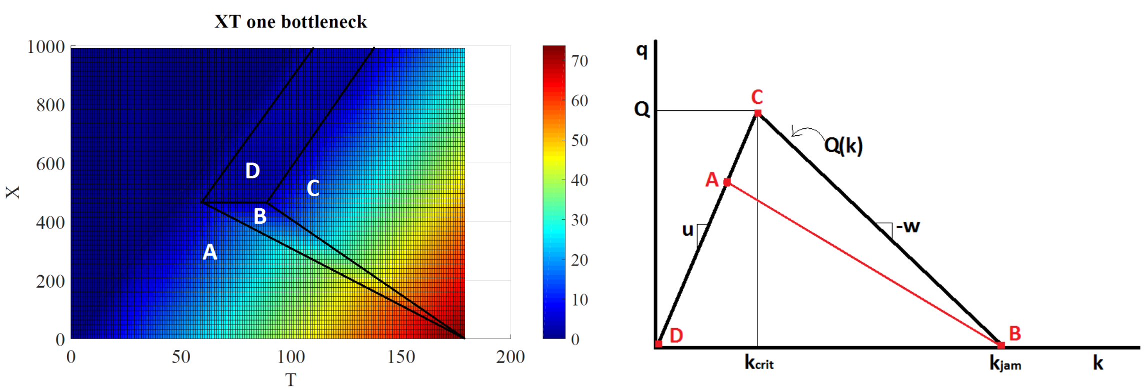
(a) Fundamental diagram for the N-model

(b) XT-diagram and iso-n lines, with 3 trajectories. The colors indicate the vehicle number.

Figure 3.2: Representation of traffic in the N-model

Shockwave Theory in the XT-plane

By combining trajectory data and the fundamental diagram, it is possible to identify traffic states. These traffic states are a combination of density and flow, for example, jam state or capacity state. Figure 3.3 shows an example the traffic states which are identified using the fundamental diagram.



(a) XT-diagram with different traffic states

(b) Fundamental diagram with different traffic states

Figure 3.3: Shockwave Theory in the N-model, with different traffic states, A = inflow, B = jam, C = Capacity, D = empty road

3.2.2 X-model

In the X-model, the fundamental diagram describes the speed V based on the spacing s. The shape of $V(s)$ is presented in Figure 3.4a and has the following characteristics:

- A speed of 0 between the origin $s = 0$ and the jam spacing s_j
- A congestion branch where speed increases with spacing. This state occurs when $s_j < s \leq s_c$, the angle of increase is equal to $\tau = 1/(wk)$
- A free flow branch where the speed is maximum and continuous (v_f). This state occurs when spacing exceeds the critical value $s > s_c$
- A critical spacing s_c which is the minimum value for spacing with maximum speed (v_f)

From the perspective of the X-model, the traffic situation of Figure 3.1 is visualized in the NT diagram 3.4b. Looking horizontally in the plot, there is a continuous increase in distance (colors) for the first vehicles, indicating that they have an undisturbed journey. Vehicle nr 20 is positioned on 1 location for a period of time, so it experiences a delay due to a jam. The length of the jam is not visible in this representation, only the delay it causes. The delay is largest for the lower vehicle numbers (but ¿5) and decreases with time for higher vehicle numbers, indicating that the jam is slowly dissolving. Furthermore, the delay starts at a distance around 400m (green color) and changes to blue colors with time, indicating that the jam is moving in the $-x$ direction.

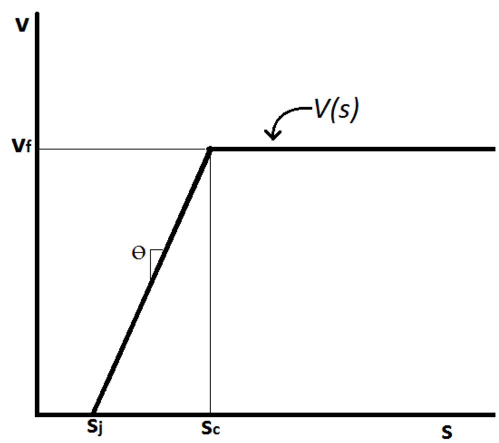
Trajectories in the X-model are iso-x lines. It tracks how many vehicles have crossed a fixed location with time, in other words, every trajectory is cumulative curve. The three indicated trajectories in Figure 3.4b are cumulative curves for approximately $x = 100, x = 400$ and $x = 600$.

From the NX diagram, also other traffic variables can be estimated. The vertical distance between two trajectories is a measure for density $\frac{\Delta n}{\Delta X} = -k = -\frac{1}{s}$ and the horizontal distance can be used to estimate the pace $\frac{\Delta t}{\Delta X} = p = \frac{1}{v}$.

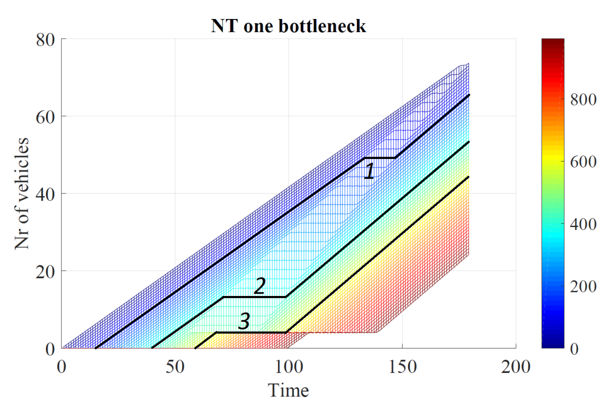
Based on the assumption that vehicles cannot disappear from the road, a continuity equation can be retrieved. The continuity equation for the X-model is:

$$\frac{\delta s}{\delta t} + \frac{\delta v}{\delta n} = 0$$

which states that the change in spacing s with time t and the change of speed v with vehicle number n should equal out to zero.



(a) Fundamental diagram for the X-model



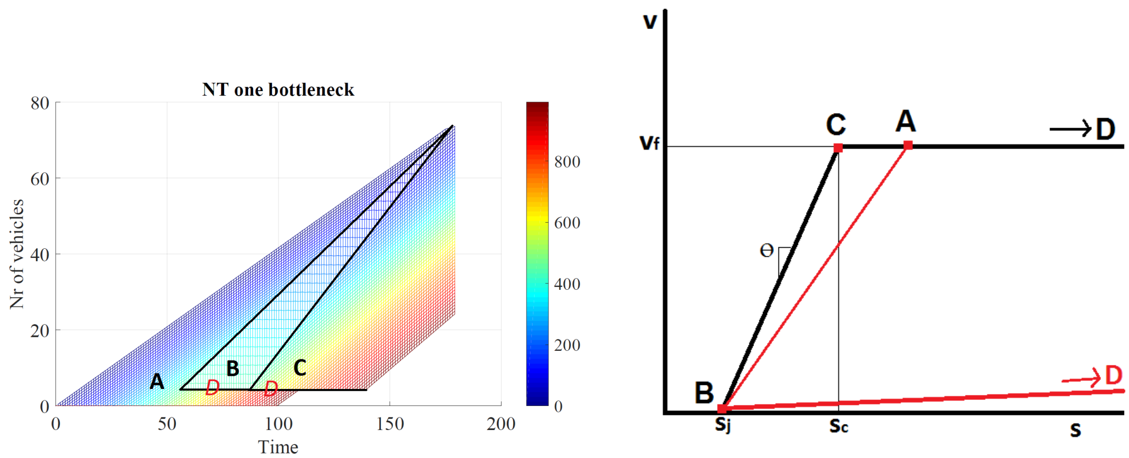
(b) NT-diagram and iso-x lines, with 3 trajectories. The colors indicate the location x.

Figure 3.4: Representation of traffic in the X-model

Shockwave Theory in the NT plane

Similar to Shockwave Theory in the N-model, it is possible to also identify traffic states in the X-model. This is done by combining the trajectory data in the NT-plane with the fundamental diagram, as shown in Figure 3.5. Four states

can be identified: Inflow A, jam B, capacity C and empty road D. The steps to identify traffic states is similar to the shockwave theory in the XT-plane, with 1 exception. The spacing on an empty road can be infinitely large, so state D does not have an exact position in the fundamental diagram. Since the spacing at D is infinitely large, the shock between jam B and empty road D is horizontal in the limit. Furthermore, the size of the shock is infinitely small, so it is invisible in the NT-diagram. The connecting line between inflow state A and jam state B, indicates how fast the jam is growing, whereas the connection line between jam state B and capacity state C, gives information on how the jam dissolves.



(a) NT-diagram with different traffic states

(b) Fundamental diagram with different traffic states

Figure 3.5: Shockwave Theory in the X-model, with different traffic states, A = inflow, B = jam, C = Capacity, D = empty road. Traffic state D is infinitely small.

3.2.3 T-model

The variables used to describe traffic states in the T-model are pace p and headway h. The characteristics of the fundamental diagram for headway as function of pace $H(p)$ are:

- There is a capacity state at a minimum pace p_c where headway is minimum H_c at $H = 1/C$
- Headway increases from H_c to infinity in the free flow branch, so it is a vertical line at $p = 1/v_f$
- Headway increases with pace in the congested branch with angle ϵ which equals $1/k_j$
- For high values of pace (=low speed), the headway increases to infinity

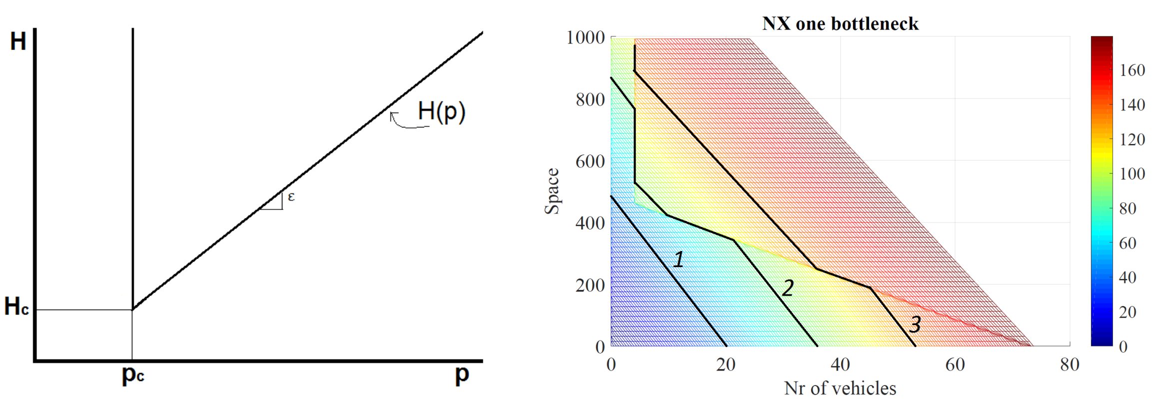
The traffic state in the T-model is represented in the NX-plane, see Figure 3.6. In this graph, the colors represent the time, so a jump in coloring indicates the boundary to a different traffic state. A horizontal line in the NX diagram gives the time at which all vehicles cross a certain location, so the vertical color jump indicates that there is an empty road. A vertical line in the NX diagram follows 1 vehicles and give the time at which that vehicle passes a certain position. The color jump in vertical direction indicates that the vehicle takes a longer time than usual before reaching the next position, so it is caught in a jam. The extent of the color jump indicates the magnitude of the jam. A large jump equals a large delay.

Trajectories in the T-model are iso-t lines which are basically snapshots. It provides the position of all vehicles on the road at a certain time step. The orientation of the line is from the upper left to the lower right, which is opposite to the trajectories in the other two models. The different orientation is a result of the convention of scale in vehicle number, as mentioned in section 3.1.1. The example trajectory nr 2 in Figure 3.6b indicates that the first 5 vehicles are situated around 800 distance, followed by an empty road. Then, approximately 5 vehicles are going at capacity flow between 550 to 450 distance, while 10 vehicles are trapped in a jam around distance 400. Finally, 10 vehicles are situated between distance 0 and 375 in the 'normal' or initial traffic state. It is difficult to see in this example, but the slope of trajectory is smallest (least negative) in the capacity state. In general, the vertical distance between two trajectories is an estimation of speed $\frac{\Delta x}{\Delta T} = v = \frac{1}{p}$ while the horizontal distance is a measure for the flow $\frac{\Delta n}{\Delta T} = q = \frac{1}{h}$.

Assuming that no vehicles disappear from the road, the continuity equation for the T-model is given by:

$$\frac{\delta p}{\delta n} - \frac{\delta h}{\delta x} = 0$$

stating that the change of pace p with vehicle number n should be equal to the change in headway h with distance x.



(a) Fundamental diagram for the T-model

(b) XN-diagram and iso-t lines, with 3 trajectories. The colors indicate the time.

Figure 3.6: Representation of traffic in the T-model

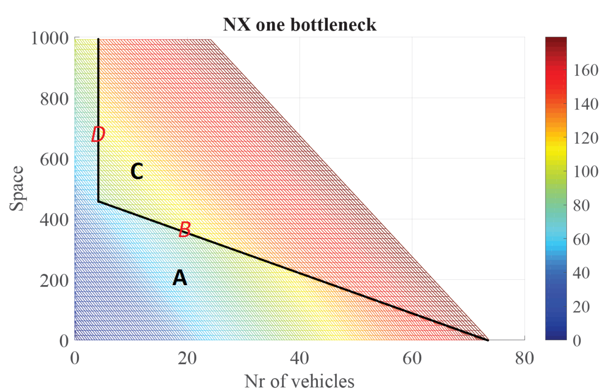
Shockwave Theory in the NX-plane

Due to the opposite orientation of the trajectories, the fundamental diagram needs to be adjusted before shockwave theory can be applied. Since the free flow branch of the $H(p)$ is vertical, it is sufficient to only reflect the congested branch in the line $H = H_c$, resulting in a line with angle $-q$. This line is partially drawn in Figure 3.7b, with jam state B' located at minus infinity.

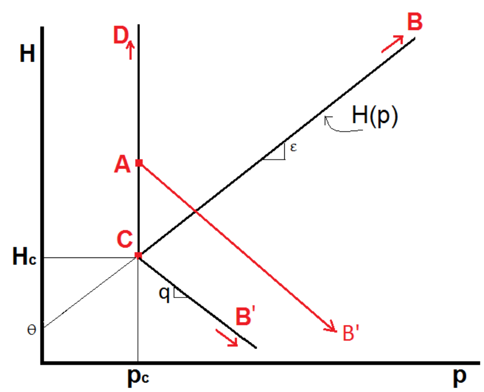
Returning to the XN-diagram, two traffic states are easily recognized due to the jumps in color, with initial state A in the lower part and the capacity state C in the upper right part. In between these states, two infinitely small traffic states exists. State D indicates the empty road, and state B indicates the jam state. The headway for an empty road and at stand still are both infinitely large, therefore the states B and D are not physical points in the fundamental diagram, but indicated with the arrows. Because the jam state is infinitely large, the shockwave of the jam (B) with the inflow (A) and outflow (C) are parallel, with an infinitely small area in between. A similar reasoning holds for the empty road state D, which is situated in between the initial state A and capacity state C.

Only the tail of the queue is visualized in the NX-diagram, and not the queue itself. This is because stopped vehicles do not move in X and therefore are not visualized. Only in the time step where they start to move again, the vehicles are visualized in the next state. The tail of the jam moves in $-x$ direction with slope $-\epsilon = -\frac{1}{k_j}$, so it depends on the jam density only. In this example, the tail of the jam grows till it reaches the beginning of the road. If the jam had dissolved sooner, there would have been another vertical shock between capacity state C and inflow state A, at the vehicle number that does not experience any delay anymore.

Although shockwave theory can be applied in this representation, it does not provide extra information. The occurrence of a jam can be identified from this graph and the growth of the jam in space can be estimated. However, it is impossible to identify when the congestion will be dissolved. This is a disadvantage of this representation.



(a) XN-diagram with different traffic states



(b) Fundamental diagram with different traffic states

Figure 3.7: Shockwave Theory in the T-model, with different traffic states, A = inflow, B = jam, C = Capacity, D = empty road. Traffic state B and D are infinitely small.

3.3 Summary

In summary, the N-model is more intuitive and easy to understand. The X-model is mathematically easier but more difficult to interpret. The X-model is difficult to grasp and does not yet have clear benefits. Table 3.2 provides an overview of the main parameters used in the different representation.

Table 3.2: Overview of parameters

	$N(x,t)$	$X(n,t)$	$T(n,x)$
1st explanatory variable	x	n	n
2nd explanatory variable	t	t	x
Independent variable	$-k$	$-s$	p
Dependent variable	q	v	h
FD	$Q(k)$	$V(s)$	$H(p)$
Trajectories	iso-n	iso-x	iso-t
slope trajectories	$\frac{dx}{dt} = v$	$\frac{dn}{dt} = q$	$\frac{dx}{dn} = -s$
vertical distance	$\frac{\Delta x}{\Delta N} = -\frac{1}{k}$	$\frac{\Delta n}{\Delta X} = -k$	$\frac{\Delta x}{\Delta T} = v$
horizontal distance	$\frac{\Delta t}{\Delta N} = \frac{1}{q}$	$\frac{\Delta t}{\Delta X} = p$	$\frac{\Delta n}{\Delta T} = q$

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Appendix A

Matlab code for creating (slanted) cumulative curves

```
function cumcurves()
%This function will give the cumulative curves for a flow which
%has (in time) three values, and one bottleneck halfway
q0=[3600;5000;2000];%the three demands
Tchange=[60;90];%times in minutes at which the demands change
c=4000;%capacity
T=0:200;%minutes
dt=1/60;%time steps (in hours: time step is 1 min)
dem=q0(end)*ones(size(T));%pre-allocate demand function to the last demand value
for(i=numel(Tchange):-1:1)
    dem(T<Tchange(i))=q0(i);%adapt the demand function
end

figure;
plot(dem,'linewidth',2)
hold on
plot(repmat(c,size(dem)),'r--','linewidth',3)
ylim([0 6000])
legend('Demand','Capacity','location','Northeast')
ylabel('Flow (veh/h)')
xlabel('Time (min)')
exportfig('Demand')
%%

Nin=dt*cumsum(dem);
qout=zeros(size(Nin));
qout(1)=dem(1);%in veh/h
queued=zeros(size(dem));
for(t=2:numel(T))
    qout(t)=1/dt*min(dt*c, dt*dem(t)+queued(t-1));
    queued(t)=queued(t-1)+dt*dem(t)-dt*qout(t);
end
Nout=dt*cumsum(qout);
figure;plot(Nin,'linewidth',2,'color',[0.5 0.5 1]);hold on;plot(Nout,'r--','linewidth',3)
legend('N_{in}','N_{out}','location','Northwest')
ylabel('Cumulative flow')
xlabel('Time (min)')
exportfig('Cumulative curves')
%%

%compute total delay:
TotalD=dt*sum(queued)%then the total delay in hours
```

```
NrVeh=Nin(end);
AvgDelay=TotalD/NrVeh%then the total delay in hours
AvgDelayMin=60*AvgDelay;%then the total delay in hours

%%
Tin=interp1(Nin,T,1:NrVeh);%time to enter for each vehicle -- interpolation
Tout=interp1(Nout,T,1:NrVeh);%time to exit for each vehicle -- interpolation
DT=Tout-Tin;%additional travel time
figure;
plot(1:NrVeh,DT,'linewidth',2)
xlabel('Vehicle number')
ylabel('Delay (min)')
exportfig('Delay per vehicle')

figure;
plot(Tin,DT,'linewidth',2)
xlabel('Entry time (min)')
ylabel('Delay (min)')
exportfig('Delay as function of time')
```