Traffic flow on pedestrianized streets

Carlos F. Daganzo and Victor L. Knoop

August 2, 2015

Abstract

Giving pedestrians priority to cross a street enhances pedestrian life, especially if crosswalks are closely spaced. Explored here is the effect of this management decision on car traffic. Since it is known from queuing theory that the closer the crosswalk spacing the lower the effect of pedestrians on cars, scenarios where pedestrians can cross anywhere are best for both cars and pedestrians. These are the scenarios studied.

Approximate formulas for key features of a street's (macroscopic) fundamental diagram are derived for different levels of pedestrian cross-flows, including tight analytic upper and lower bounds for the street's capacity. The formulas reveal that pedestrian crossings: (i) reduce capacity by an amount proportional to the square root of the pedestrian flow; and (ii) increase, both, the free-flow and backward-wave paces by amounts proportional to the pedestrian flow.

1 Introduction to the problem

In urban environments, traffic flow is affected by the external influence of pedestrians. If pedestrians are regulated by traffic lights the only disruptions to flow are the traffic light themselves. This situation is simple and formulas to predict delay already exist; see e.g. Daganzo (1997). Therefore this paper focuses on the unsignalized case. The subcase in which cars have priority over pedestrians is not interesting because (i) pedestrians have no effect on traffic flow and (ii) the ensuing pedestrian delays have already been described with queuing theory (Tanner, 1951). Therefore the focus is narrowed to the subcase in which pedestrians have priority at all crossings.

We want to understand the effect of these crossings on the traffic stream. The effect of a single crosswalk is already well understood. Queuing formulas, in which the cars are customers served by the crosswalk, exist for both the street capacity and the expected traffic delay (Hawkes, 1965, 1968, Daganzo, 1977). These formulas predict that splitting the pedestrian flow of a single crosswalk among several widely separated crosswalks always increases the street's capacity and reduces car delay; i.e., that a street design with crosswalks every 25m is better for cars than one with crosswalks every 100m. We conjecture that this continues to be true if the crosswalk separation tends to zero; i.e., if we allow pedestrians to cross anywhere. Since crossing anywhere is also good for pedestrians, it is probably the best thing to do if pedestrians are to have priority. This pedestrianization shall be the scenario considered in this paper.

A street with multiple pedestrian crosswalks can be modeled as a serial queuing system. However, when the crosswalks are very closely spaced, car queues will spill back over upstream crosswalks, "blocking" service. Unfortunately, queuing theory does not provide easy answers to problems with spillbacks – the solution with only two servers is already very complicated; see Newell, (1979). For this reason, our crossing-anywhere scenario will not be studied here with the tools of queuing theory, but with a combination of symmetry arguments, dimensional analysis, simulation and traffic flow theory.

We shall consider an infinitely long homogeneous street with cars and crossing pedestrians. It is assumed that cars behave according to the kinematic wave theory of traffic flow (Lighthill and Witham, 1955, Richards, 1956), and that the fundamental diagram relating flow q, and density k, is triangular as proposed in Newell (1997). The fundamental diagram (FD) relationship is denoted $q = Q(k)$. As is well known, kinematic wave theory is equivalent to two other representations of traffic that will be used in this paper: (i) the variational theory of traffic flow with a linear cost function (Daganzo, 2005, 2005a, 2006); and (ii) Newell's simplified car-following model (Newell, 2002). Pedestrian arrivals are assumed to be homogeneous in space, stationary in time and mutually independent. Each pedestrian is assumed to interrupt traffic for a fixed amount of time, which is equal for all pedestrians. We are interested in seeing how these random interruptions modify the macroscopic fundamental diagram (MFD) of the street, and in particular how much they reduce the street's capacity and its free-flow speed.

To answer these questions the paper is organized as follows. Section 2 shows that the MFD can be expressed as a function of only two parameters and that the capacity can be expressed as a function of a single one; i.e., that a single curve yields the capacity for all scenarios. Section 3 examines this curve with simulations and presents an analytical approximation. Building on these results, Sec. 4 then presents simulated results and approximate analytic formulae for the MFD. Finally, Sec. 5 presents some conclusions

2 Simplifications

The problem in question is finding the FD of a pedestrianized street along the lines we have described. To define an instance of the problem one needs to characterize the street and the pedestrians. Since the street has a triangular FD, three parameters suffice to describe it. We shall use: (i) the street capacity without pedestrians, q_o; (ii) the jam density k_j; and (iii) the optimum density, k_o. Other FD features can be derived from these three parameters. This paper will use: (a) the free-flow speed, $v_f = q_o/k_o$; (b) the backward wave speed, $w = q_0/(k_j - k_o)$; and (c) and the flow-intercept of the congested branch, $r = k_j q_0/(k_j - k_o)$.

To describe the pedestrians two parameters suffice. We shall use: (iv) their arrival flux f in pedestrians per unit time per unit length of street; and (v) the time, τ, that each pedestrian blocks the street. Thus, an instance of the problem is defined by five parameters in total.

To simplify the formulation we shall work from now on in a system of units for time, distance and vehicle number (u_t, u_x, u_n) such that the values of τ, q_o and k_j equal 1. The reader can verify that this is always possible by choosing $(u_t \equiv \tau, u_n \equiv q_o\tau, u_x \equiv q_o\tau/k_j)$. For example, if $q_o = 1800$ v/h, $k_j = 200$ v/km and $\tau = 5$ s then $u_t = 5$ s, $u_n = 2.5$ v, $u_x = 12.5$ m. Thus, from now on and without any loss of generality: $\tau = q_o = k_j = 1$ so that these parameters are eliminated.

The remaining two parameters define the problem. The optimum density, $k_o \in [0, 1]$, characterizes the street, and the pedestrian flux, $f \geq 0$, the pedestrians. The latter can be interpreted as the expected number of pedestrian arrivals to a section of street of length u_x (comparable with a few car spacings) during a time τ (consisting of a few seconds). In most real applications arrivals in such a narrow window should be rare so that typically $f << 1$. If this is not the case, pedestrian traffic lights are a better option since the disruption they impose on both cars and pedestrians is largely independent of f.

The foregoing means that the FD solution for all problems can be expressed as a two-parameter family of curves. This simplification will be exploited in Sec. 4. Interestingly, it also turns out as is shown in Sec. 2.1 below that k_o does not influence the maximum FD flow, i.e., the street capacity under pedestrian interruptions $\boldsymbol{q_o}$. (Boldface shall be used from now on for variables representing pedestrianized conditions.) This result is useful because it establishes that a single curve, $\boldsymbol{q_o}(f)$, describes the capacities of all problem instances. This simplification will be exploited in Sec. 3.

2.1 Invariance of the street capacity with respect to k_o

To start, let us define two technical terms: (i) "pedestrian realization", or "realization" for short: a randomly drawn set of pedestrian arrival points in the time-space plane, $\{..., (t_i, x_i), ...\}$; and (ii) "conditional capacity": the street's maximum flow possible for a given pedestrian realization. Note that the street's capacity $\boldsymbol{q_o}$ we seek is the average of the conditional capacities of an infinite number of random pedestrian realizations.

To show that $\boldsymbol{q_o}$ does not depend on k_o, we shall show that the value of $\boldsymbol{q_o}$ for an arbitrary $k_o \in [0, 1]$ is the same as for the case with $k_o = 1/2$. We will do this by showing that random pedestrian realizations for these two cases can be generated in pairs that have the same conditional capacities. Hence the average of the conditional capacities for the two cases is the same.

To generate these realization pairs we shall use a linear change of variable for the time coordinate, $t \to t'$, whereby new clocks at every location are started at the moment of passage of a moving observer that travels with a constant pace, u:

$$t' = t - x/u. \quad (1)$$

As the reader can verify, this change of variable leaves invariant the flow but changes the speed and density variables as follows:

$$1/v' = 1/v - 1/u. \quad (2)$$

and

$$k' = k - Q(k)/u. \quad (3)$$

The pace of the observer is chosen to be $1/u = k_o - 1/2$ because then, as shown by (3), $k'_o = k_o - Q(k_o)(k_o - 1/2) = k_o - (k_o - 1/2) = 1/2$. In addition, note from (2) that the free-flow and backward wave speeds with $u = k_o - 1/2$ become equal and of opposite sign, since $1/v'_f = 1/v_f - (k_o - 1/2) = k_o - k_o + 1/2 = 1/2$ and $1/(-w') = 1/(-w) - (k_o - 1/2) = (k_o - 1) - k_o + 1/2 = -1/2$. Thus, the transformation converts a street with arbitrary k_o as shown in Fig. 1a, into one with symmetric limiting speeds and $k'_o = 1/2$ as shown in Fig. 1b. We shall call the problem and the FD as originally formulated "primal", and the ones arising from the change of variable "dual".

The transformation also converts any set of points in the primal time-space plane (t, x) into a unique image in the dual plane, (t', x), and viceversa – i.e., the transformation is a bijection. In particular, the transformation associates with every (primal) draw of a pedestrian realization a unique dual image, as illustrated by Fig. 1. Furthermore, since the transformation is linear, it preserves for these dual images the defining properties of primal pedestrian arrivals; i.e., their homogeneous, stationary and independent nature; and their arrival flux, f. Thus, the dual images of the random primal realizations can be interpreted themselves as random realizations of a dual problem of the same type as the original, albeit with a transformed FD.

The only difference between the primal and the dual problems is that the optimum density of the dual is $k'_o = 1/2$ instead of k_o. Thus, our assertion shall be proven if we can show that the average of the conditional capacities across all dual realizations (for the case with $k_o = 1/2$) is equal to the average of the primal conditional capacities (for the case with $k_o \in [0, 1]$). This equality is proven by the proposition below, which actually establishes something stronger: that the conditional capacities of a primal realization and its dual image are always equal.

In the proof of the proposition and elsewhere in this paper we shall model the pedestrian interruptions of a given realization (primal or dual) as fixed bottlenecks with zero capacity, each pinned at x_i and lasting from t_i (or t'_i) to $t_i + 1$ (or $t'_i + 1$); and we shall use VT to evaluate the street's conditional capacity given the bottlenecks. The following definitions and facts from VT specific to the conditions of our paper will be used – see Daganzo (2005, 2005a) for justifications of the facts:

- Definition 1: A path is "valid" if the slope between any two of its points is in $[-w, v_f]$.

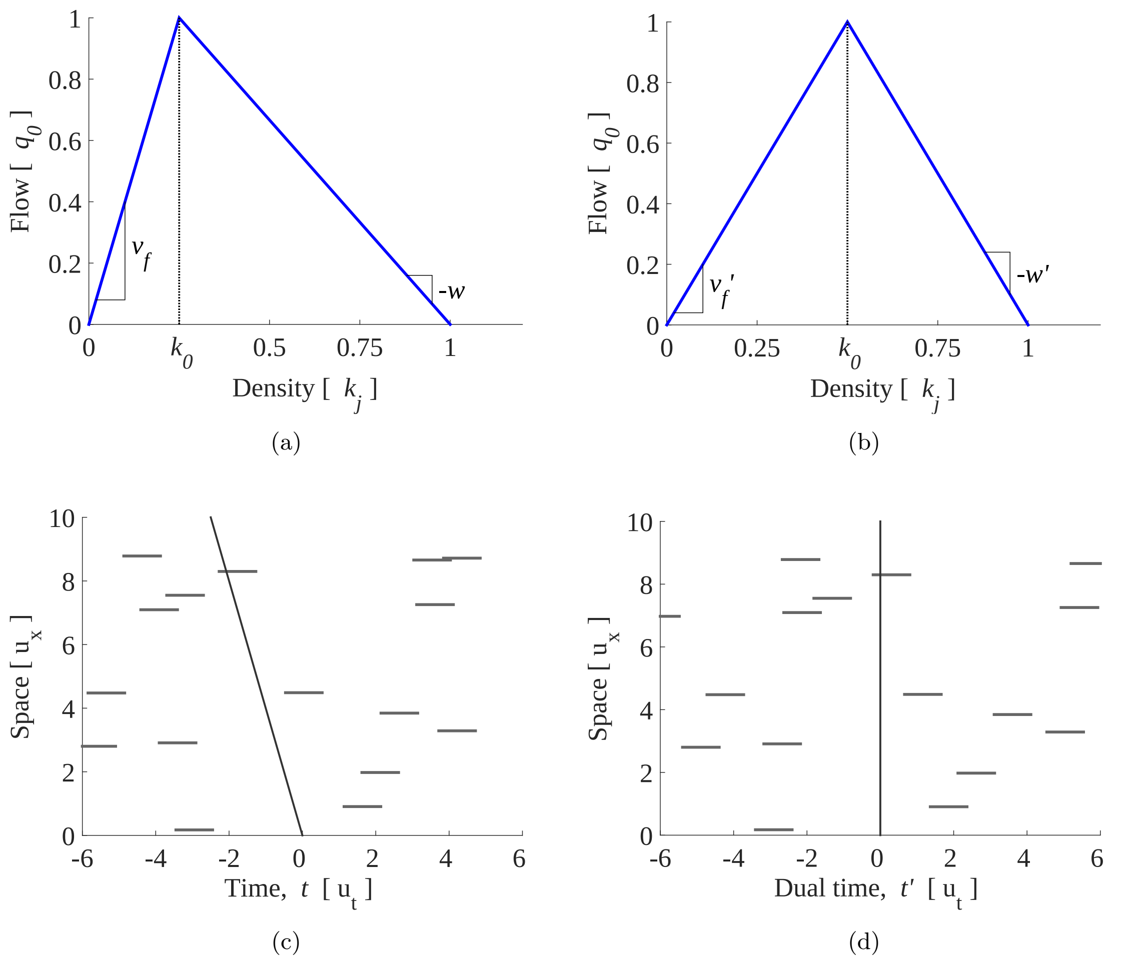


Figure 1: The primal-dual transformation [symbols in brackets are the measurement units]: (a) A primal FD; (b) the symmetric dual FD; (c) a primal pedestrian realization with the observer's trajectory; (d) dual image of the primal realization and the transformed observer trajectory

- Definition 2: Slopes equal to v_f or $-w$ are called "extremal".
- Definition 3: A path is "feasible" if it is, both, valid, and goes from the origin to a point $(T, 0)$ on the time axis. The time T is the path's "duration".
- Fact 1: The conditional capacity is the least cost of all feasible paths with $T \to \infty$.
- Fact 2: The cost per unit time of a path segment with extremal slope v_f is 0.
- Fact 3: The cost per unit time of a path segment with slope 0 (not on a bottleneck)

is $q_o = 1$.

- Fact 4: The cost per unit time of a path segment with extremal slope $-w$ is the flow-intercept of the FD's congested branch, $r = 1/(1 - k_o)$.
- Fact 5: If a path segment is entirely on a bottleneck its cost rate is 0.
- Fact 6: Without bottlenecks, all valid paths joining two points have the same cost.

Proposition 1. *A primal realization and its dual image have the same conditional capacity.*

Proof. We need to show that the minimum cost across all feasible paths with $T \to \infty$ is the same for the original and transformed problems. To establish this, it suffices to show that for every T: (i) our transformation establishes a bijection between the primal and dual sets of feasible paths; and (ii) the cost of every feasible primal path is the same as that of its (also feasible) dual image. If this happens, all least cost paths occur in primal/dual pairs with identical costs, and therefore the primal and dual conditional capacities are the same.

Assertion (i) is true because (a) the transformation maps the origin and point $[T, 0]$ onto themselves, as can be seen from from Eq.(1); and because (b) images and pre-images of feasible paths are feasible (as can be seen from (2) which implies $\{v \in [-w, v_f] \Leftrightarrow v' \in [-w', v'_f]\}$).

To complete the proof, assertion (ii) is now established. To this end, consider a feasible primal path $\mathcal{P}$ and its feasible dual image $\mathcal{P}'$; and let O and O' be the total combined times that each of these paths overlaps with bottlenecks. Now note from (1) that every segment of the primal time-space plane with constant x (such as any overlap between $\mathcal{P}$ and a bottleneck) is mapped onto a dual segment with constant x of the same duration. Obviously then, the total overlap durations of $\mathcal{P}$ and $\mathcal{P}'$ should be the same; i.e., $O \equiv O'$.

To finish proving assertion (ii) and conclude the proof it is now shown that the cost of any feasible path (primal or dual) is the amount of time it does not overlap with bottlenecks – this proves the assertion because then the cost of $\mathcal{P}$ is $[T - O] \equiv [T - O']$, which is the cost of $\mathcal{P}'$. The cost of a feasible path is first calculated as if there were no bottlelecks. In this case, as per VT Fact 6 mentioned above, the path's cost equals the cost of the zero-slope valid path between the origin and the path's end-point point $[T, 0]$. This cost is T as per Fact 3. So the path's cost without bottlenecks is T. Now consider the portion of the path that overlaps with bottlenecks. Bottlenecks reduce the cost of this portion to 0, as per Fact 5, whereas the cost without bottlenecks was the duration of the overlap, as per Fact 3. Since the path's cost without bottlenecks is the total time T, and the introduction of pedestrian bottlenecks reduces this cost by the amount of overlap, it follows that the final cost is the amount of time the path does not overlap with bottlenecks, which is the same for the primal and dual images. □

This proposition establishes, not just that the conditional capacity for any realization is independent of k_o but that this independence also holds for the street's capacity $\boldsymbol{q_o}$. Thus, $\boldsymbol{q_o}$ is a function of a single variable: $\boldsymbol{q_o}(f)$. The next section characterizes this function.

3 Results: the street capacity as a function of pedestrian flux

The function $\boldsymbol{q_o}(f)$ is estimated with a mix of simulations and analysis. The simulation results are described in Sec. 3.1, and the analysis in Sec 3.2.

3.1 Simulation results

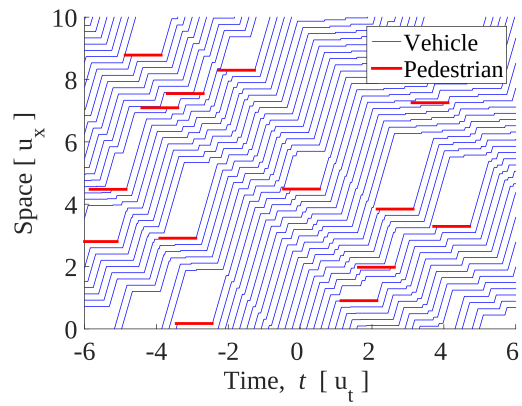
A pedestrianized street was simulated using Newell car-following model because this model is equivalent to the KWT and allows for the introduction of pedestrians. Recall that in this model, a car advances in each increment of time Δt to the most advanced position that does not exceed: (i) its current one $x(t-\Delta t)$ plus the maximum distance it can travel in Δt, $v_f\Delta t$; and (ii) the leader's position x^L, evaluated $1/r$ seconds before t, minus one jam spacing, $1/k_j$. Pedestrian interruptions are added by including as a third bound (iii) the position $x^P(t)$ of the closest pedestrian bottleneck at or downstream of $x(t)$ at time t. The rule with pedestrians is:

$$x(t) = \min\{x(t-\Delta t) + v_f\Delta t; x^L(t-1/r) - 1/k_j; x^P(t)\} \tag{4}$$

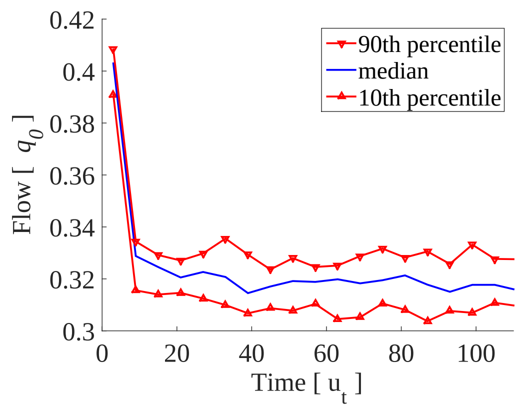
For the model to be accurate, Δt should be small compared with both the duration of a pedestrian interruption, τ, and $1/r$. We set $\Delta t = 0.1$s because in our battery of experiments, $\tau = 10$s and $1/r > 2$s. Figure 2a shows an example set of trajectories for one of the simulation runs.

The simulation was implemented for a circular street as this allowed us to to use a fixed number of vehicles and control for density. In all instances the simulation was started with the vehicles evenly distributed on the street and traveling at the equilibrium speed. Pedestrians were then activated after a time slightly greater than $1/r$ to ensure that rule (4) could be applied – note the right side of (4) involves a time lag of duration $1/r$. The flow in each time window was then evaluated with Edie's recipe (Edie, 1963) as the ratio of the total distance traveled by all cars in the time window and the product of the time-window duration and the street length.

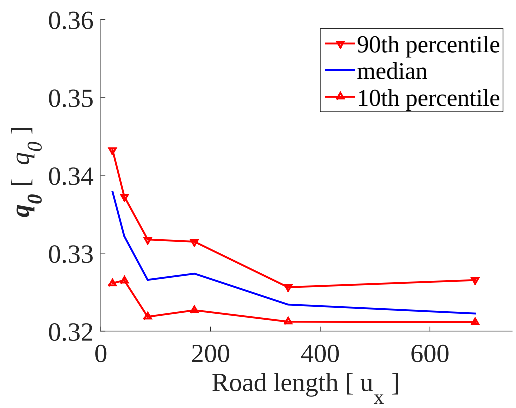
In all cases simulated the system transitioned into equilibrium in well under 3 minutes. For this reason each simulation run was made to last 20 minutes and its first 10 minutes were discarded. Figure 2b shows these initial effects for one of the simulated scenarios. It was also found that the street flow decreases with the street length L, and stabilizes once the street is several kilometers long; see Fig. 2c, which shows another simulated scenario. Since in all cases a stable flow was found for $L \geq 15$ km we chose $L = 15.36$ km to simulate an infinite street.



(a)



(b)



(c)

Figure 2: Car-following model features [symbols in brackets are measurement units]: (a) A set of simulated vehicle trajectories for a pedestrian realization; (b) time-series of the street's flow showing the initial transient; (c) the effect of street length on flow.

A total of 36 scenarios were constructed by fixing some parameters of the problem and varying others. These selections resulted in 4 different values of the dimensionless critical density $k_o = \{0.2; 0.3; 0.5; 0.75\}$ and 9 values of the dimensionless pedestrian flux, $f = \{0.001; 0.002; 0.005; 0.01; 0.02; 0.05; 0.1; 0.2; 0.3\}$. Thus, a total of 36 scenarios were simulated. For each of these scenarios, the average system flow was evaluated for 40 different densities uniformly distributed in the range $(0, k_j) \equiv (0, 1)$; and the highest resulting flow was selected as the estimate of $\boldsymbol{q_o}$.

Figure 3 displays on the $\boldsymbol{q_o}$ vs f plane the simulated results. These are expressed by means of four solid piecewise linear curves. Each of these depicts the 9 simulated results

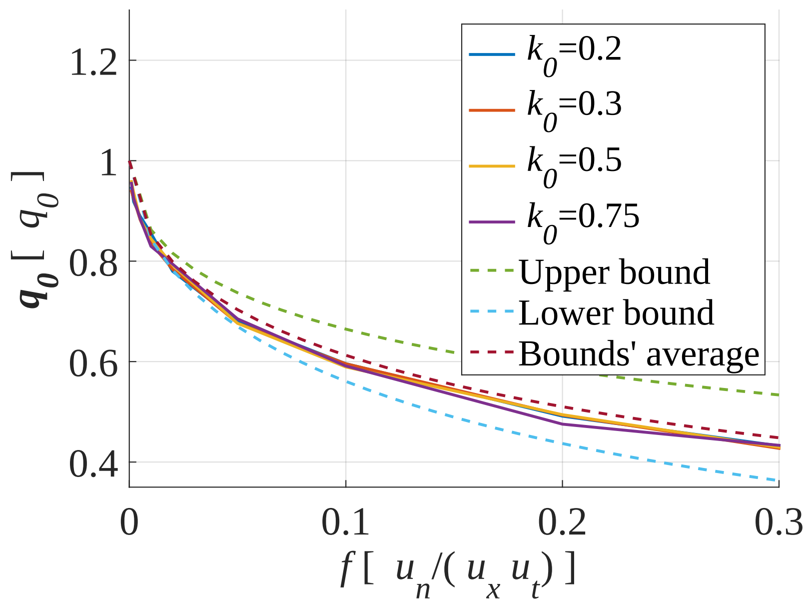


Figure 3: Capacity as function of the pedestrian flux. Symbols in brackets are measurement units.

for one value of k_o. Note how, as expected, the lines cluster together since k_o has no effect on capacity. The figure also contains three dashed curves, which are analytical bounds and an approximation. These are derived below.

3.2 Analytical bounds and an approximation

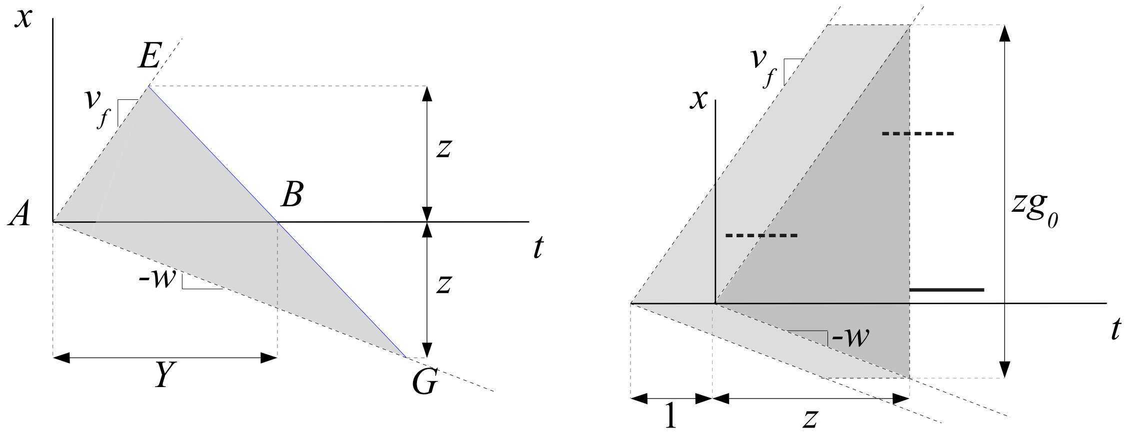
3.2.1 Upper bound

An upper bound is given by the expected cost of any feasible path. Therefore, an algorithm that yields reasonably cheap piecewise linear paths and allows the expected cost to be expressed analytically is used to produce the bound.

The algorithm is recursive. Each step involves one pedestrian, which is used to produce two consecutive linear segments. The first of these goes to the pedestrian's arrival point, and the second overlaps the pedestrian's bottleneck for its full duration, $\tau = 1$.

The pedestrian is chosen with two objectives in mind: ensuring that the path returns to the horizontal axis with probability 1 (so it is feasible for $T \to \infty$); and ensuring that the first linear segment is of short duration (since the the cost of a feasible path is the proportion of time that it does not overlap with bottlenecks).

Figure 4a illustrates how the pedestrian is chosen. The origin in that figure is at the endpoint of the previous step. Consider triangle AEG, which is defined by two rays with



(a) (b)

Figure 4: Geometrical regions used to construct the capacity bounds: (a) Upper bound; (b) lower bound – horizontal segments are hypothetical pedestrian bottlenecks..

extremal slopes emanating from the origin A and truncated at those points E and G where the ordinates are $+z$ and $-z$.

Now let z^* be the ordinate of the smallest triangle containing a single pedestrian arrival and let this be the chosen pedestrian. Note that such an arrival will be on side $\overline{EG}$ and that for a given z^* all points of this side are equally likely.

This means that the step's vertical jump is uniformly distributed in $[-z^*, z^*]$, and that its mean is zero. Thus, the sequence of ordinates forms a null-recurrent random walk that returns to zero as $T \to \infty$ with probability 1; see e.g., Feller (1968). In other words, our recursive rule produces feasible paths.

The even distribution of points on side $\overline{EG}$ also means that the expected duration of our first linear segment conditional on z^* is the duration of the horizontal segment from the origin to the midpoint, B, of $\overline{EG}$. Simple geometrical calculations reveal that this duration, which we denote Y, is:

$$Y = {}^1\!/\!_2 \, (z^*/v_f + z^*/w) = {}^1\!/\!_2 \, (z^* k_o + z^*(1 - k_o)) = z^*/2. \tag{5}$$

The expected duration $E(Y) \equiv E(z^*/2)$ is obtained from the distribution of z^*. Given that the area $A(z^*)$ of the triangle is $A(z^*) = Y z^* = z^{*2}/2$, and that the number of points in a triangle of area A is Poisson-distributed with mean fA, we have:

$$\Pr\{z^* > z\} = \exp(-fA(z^*)) = \exp(-{}^1\!/\!_2 f z^{*2}). \tag{6}$$

It follows that

$$E(z^*/2) = \frac{1}{2}\int_0^\infty \exp(-{}^1\!/_2 f z^{*2})dz = [\pi/(8f)]^{1/2}. \tag{7}$$

This is the expected duration of the non-overlap segment in one step of the recursion. Since the duration of the overlap segment is 1, we have that the expected fraction time without overlap for each step of the recursion and therefore for the whole path (i.e., our upper bound $\boldsymbol{q_o^U}$) is:

$$\boldsymbol{q_o^U} = \frac{[\pi/(8f)]^{1/2}}{1+[\pi/(8f)]^{1/2}} = \frac{1}{1+(8f/\pi)^{1/2}}. \tag{8}$$

This is the equation of the upper dashed curve in Fig. 3. For small values of f the above simplifies to:

$$\boldsymbol{q_o^U} = 1 - \sqrt{8f/\pi} + o(\sqrt{f}), \qquad \text{as } f \to 0. \tag{9}$$

3.2.2 Lower bound

Recall that the conditional capacity for a given set of bottlenecks is the least cost of a feasible path, and that such a path is composed of a sequence of (linear) segments that alternate overlapping and not overlapping with bottlenecks. If we designate by Y_i the duration of the i^{th} non-overlapping segment and by O_i the duration of the following overlapping segment then the conditional capacity (i.e., the fraction of time without overlap) can be expressed as: $\sum Y_i/(\sum O_i + \sum Y_i)$.

Since $\boldsymbol{q_o}$ is the average of $\sum Y_i/(\sum O_i + \sum Y_i)$ across realizations, it is bounded from below by any quantity that bounds from below every realized instance of $\sum Y_i/(\sum O_i + \sum Y_i)$. One such bounding quantity, denoted $\boldsymbol{q_o^L}$, is constructed below by replacing every O_i in this expression by an upper bound and every Y_i by a lower bound.

Since $O_i \leq 1$, the upper bound chosen for O_i is 1. And, since Y_i is the duration of a valid non-overlapping segment to another bottleneck, the lower bound chosen for Y_i is the duration L_i of a valid segment that reaches another bottleneck in the least time possible . Thus, our lower bound is: $\boldsymbol{q_o^L} \equiv \sum L_i/(\sum 1 + \sum L_i) = \overline{L}/(\overline{L}+1) = 1/(1+1/\overline{L})$.

A formula for $\overline{L}$ is now derived. Since our pedestrian arrival process is time- and space-independent, and ergodic, $\overline{L}$ can be evaluated assuming that the segment's starting point is at the origin. Thus, $\overline{L}$ is the expected duration of a valid segment from the origin that reaches a bottleneck in the least time possible.

As in Sec. 3.2.1, the expectation is derived starting with an expression similar to (6) for the complementary cumulative distribution function of L_i, $\Pr\{L_i > z\}$. To this end, consider Fig. 4b and note that the condition $L_i > z$ is satisfied if and only if the shaded region of the figure does not contain any pedestrain arrivals. This is true because as the figure clearly shows the bottlenecks of pedestrians arriving outside this region can only be reached in a time greater than z and those of pedestrians arriving inside can be reached in time less than z – in the lighter-shaded area the bottleneck can only be reached after the pedestrian's arrival. Now, denote by $A_L(z)$ the area of the shaded region, and note from the figure's geometry that:

$$A_L(z) = g_o z + g_o z^2/2, \tag{10}$$

where

$$g_o \equiv 1/[k_o(1-k_o)] > 0. \tag{11}$$

Thus, we can write:

$$\Pr\{L_i > z\} = \exp(-f A_L(z)) = \exp(-f[g_o z + g_o z^2/2]), \tag{12}$$

and

$$\overline{L} \equiv E(L_i) = \int_0^\infty \exp(-f[g_o z + g_o z^2/2])dz. \tag{13}$$

Now reorganize the exponent of (13) to read $-f[g_o z + g_o z^2/2] = -1/2[fg_o(z+1)^2 - fg_o]$ and introduce the change of variable, $fg_o(z+1)^2 \equiv y^2$ to find:

$$\overline{L} = \frac{\exp(fg_o/2)}{\sqrt{fg_o}} \int_{\sqrt{fg_o}}^\infty \exp(-1/2 y^2)dy = \left[2\pi \frac{e^{fg_o}}{fg_o}\right]^{1/2} \left[1 - \Phi(\sqrt{fg_o})\right]. \tag{14}$$

The last equality expresses the Gaussian integral using the standard normal cumulative distribution function, Φ.

Finally, note that our lower bound, $\boldsymbol{q_o^L} = 1/(1 + 1/\overline{L})$, increases with $\overline{L}$ and that, as per (14), $\overline{L}$ decreases with g_o, which itself depends on k_o as per (11). Since the value of k_o, can be chosen at will because the capacity $\boldsymbol{q_o}$ is independent of k_o, the tightest bound is obtained by choosing the value of k_o that minimizes g_o and therefore maximizes $\overline{L}$, which maximizes $\boldsymbol{q_o^L}$. This happens for $k_o = 1/2$, which yields $g_o = 4$. The resulting lower bound is therefore:

$$\boldsymbol{q_o^L} = 1/\big(1 + 1/\{\overline{L}\}_{g_o=4}\big) = 1/\left(1 + \left[\frac{2f}{\pi e^{4f}}\right]^{1/2} \left[1 - \Phi(2\sqrt{f})\right]^{-1}\right). \tag{15}$$

This is the expression of the lower dashed curve in Fig. 3. For small values of f the above can be shown to equal:

$$\boldsymbol{q_o^L} = 1 - \sqrt{8f/\pi} + o(\sqrt{f}), \qquad \text{as } f \to 0, \tag{16}$$

which matches expression (9) for the upper bound.

3.2.3 Approximation

Since (9) and (16) match, both of these expressions are an asymptotically exact approximation for the street capacity as $f \to 0$. Note from both expressions that the reduction in capacity caused by pedestrian flow is proportional to $\sqrt{f}$. This implies that an increase in pedestrian flow has more of an effect when the pedestrian flow is low than when it is high.

An approximation for arbitrary f, is the average of the capacity bounds (8) and (15):

$$\boldsymbol{q_o} \approx (\boldsymbol{q_o^U} + \boldsymbol{q_o^L})/2. \tag{17}$$

A numerical comparison of (8) and (15) for $f \in [0, \infty]$ reveals that $\boldsymbol{q_o^U} - \boldsymbol{q_o^L} \leq 0.22$ across all f. Thus, the potential worst-case error in (17) is 11%. Fortunately, the actual error in the relevant range that was simulated, $f \in [0, 0.3]$, is considerably smaller. Figure 3 shows that in this range, Eq.(17) very slightly but systematically overpredicts the simulated values – by about 2% in absolute terms and by about 4% in relative terms. Since this overprediction is quite steady, it can be reduced by giving the lower bound slightly more weight. We find that by giving the lower bound 66% of the weight, the differences between the simulation and the analytical estimates are about 1%.

4 The macroscopic fundamental diagram

This section provides approximations to the MFD, both simulated and analytical.

4.1 Simulation results

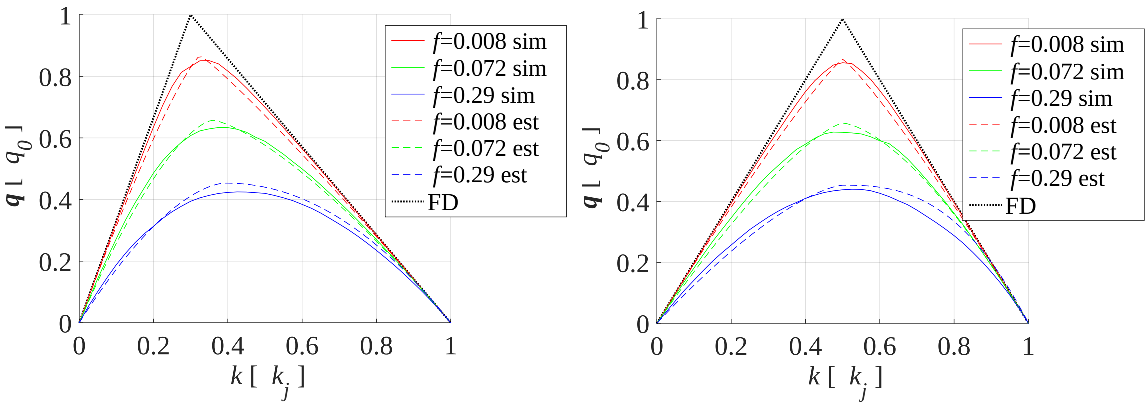
Unlike the street capacity, which depends only on f, the the fundamental diagram of the pedestrianized street, $\boldsymbol{Q}(k)$ depends on both f and k_o. For this reason we simulated six scenarios for three values of f and two values of k_o. Although the pedestrian flux, f, can change by an order of magnitude or more, it is usually small. Thus, most facilities with non-zero pedestrian flux should exhibit a value that is within a factor of three of one of the following: $f = \{0.008, 0.072, 0.29\}$. For pedestrianized streets low speed limits are likely, so values of k_o in the range $[0.25, 0.6]$ are likely. For this reason, we considered $k_o = \{0.3, 0.5\}$.

Figures 5a and 5b depict by means of solid lines the 3 MFD's corresponding to the mentioned values of f for $k_o = 0.3$ and 0.5, respectively. As a point of comparison, the figures also include the FD's for $f = 0$. Flow estimates for other values of f and k_o can be obtained by interpolation. Alternatively, the approximate analytical results of the next section, which are shown by solid lines in the figure, can also be used.

4.2 Analytical results

Analytic formulas for the slopes of $\boldsymbol{Q}(k)$ for $k = 0$ and $k = 1$ are derived first. These slopes are the pedestrianized free-flow speed $\boldsymbol{v_f}$, and the backward wave speed $-\boldsymbol{w}$. The formulas are then used to construct the MFD approximation.

To estimate $\boldsymbol{v_f}$ consider the expected time required by an isolated car to travel a distance $\Delta x \to 0$. This time is $\Delta x / v_f = \Delta x k_o$ if the car is not interupted by a pedestrian. An interruption happens if a pedestrian arrives in the window of space-time preceding



(a) k_0=0.3

(b) k_0=0.5

Figure 5: Estimated MFD's for two values of k_o and three values of f. Symbols in brackets are the measurement units. Solid lines are simulated results and dotted lines analytical estimates.

the car by a time duration $\tau = 1$, and the average duration of any such interruption is $\tau/2 = 1/2$. Since the pedestrian flux is f and the window size is $\Delta x \to 0$, an interruption occurs with probability $f\Delta x$. For $\Delta x \to 0$ the probability of multiple interruptions is $O(\Delta x^2)$. Thus, since $\Delta x \to 0$, the expected delay added by interuptions is $f\Delta x/2$. Hence, interruptions add $f/2$ units to the vehicle's pace so that the free-flow pace is:

$$1/\boldsymbol{v_f} = 1/v_f + f/2 = k_o + f/2. \tag{18}$$

To estimate $-\boldsymbol{w}$ we consider a model of empty spaces or "holes". We imagine that at all times the street is uniformly filled to jam density with a combination of vehicles and "holes". For this to happen, when a vehicle advances past an observer, a hole must flow in the opposite direction, and the sum of the densities of holes and vehicles must be k_j.

So, if we use tildes to denote the flow, speed and density variables associated with holes, we have that: $k + \widetilde{k} = k_j = 1$ and $q = -\widetilde{q}$. In a steady state, it must be true that $\widetilde{q} = -q = -Q(k) = -Q(k_j - \widetilde{k})$; i.e., there is a FD relation between $\widetilde{k}$ and $\widetilde{q}$, $\widetilde{Q}(\widetilde{k}) = -Q(k_j - \widetilde{k})$. Consideration also shows that if flow of cars is interrupted by a bottleneck, so is the flow of holes. Thus, holes obey the exact same dynamics as cars; the only difference is that their FD and MFD are related by our change of variable; i.e.:

$$\widetilde{Q}(\widetilde{k}) = -Q(k_j - \widetilde{k}) \qquad \text{and} \qquad \widetilde{\boldsymbol{Q}}(\widetilde{k}) = -\boldsymbol{Q}(k_j - \widetilde{k}). \tag{19}$$

Consideration of (19) shows that $\widetilde{v_f} = -w$ and $\widetilde{\boldsymbol{v_f}} = -\boldsymbol{w}$. We have already shown that $1/\boldsymbol{v_f} - 1/v_f = f/2$ for cars. Since holes satisfy the same model as cars, it is also true that

$1/\widetilde{\boldsymbol{v_f}} - 1/\widetilde{v}_f = f/2$. And since $\widetilde{v}_f = -w$ and $\widetilde{\boldsymbol{v_f}} = -\boldsymbol{w}$, it follows that $-1/\boldsymbol{w} + 1/w = f/2$ so that:

$$-1/\boldsymbol{w} = -1/w + f/2 = -1 + k_o + f/2. \tag{20}$$

In summary, we see from (18) and (20) that the exitence of pedestrians increases the two extremal paces by the same amount, $f/2$. This can be roughly seen in Fig. 5. We are now ready to construct the approximation.

We start with a trapezoidal upper bound. Since the MFD $\boldsymbol{Q}(k)$ is concave, an upper bound to it is given by the trapezoid formed by the lines: $\boldsymbol{q} = \boldsymbol{v_f}k$, $\boldsymbol{q} = \boldsymbol{q_o}$ and $\boldsymbol{q} = \boldsymbol{w}(1-k)$; i.e. by the right side of:

$$\boldsymbol{q} \leq \min\{\boldsymbol{v_f}k\ ;\ \ \boldsymbol{q_o};\ \ \boldsymbol{w}(1-k)\}, \tag{21}$$

where $\boldsymbol{v_f}$ and $\boldsymbol{w}$ are given by (18) and (20), and $\boldsymbol{q_o}$ can be approximated by (17).

The proposed approximation is a smooth concave curve that is tangent to the trapezoid (21) at its two extremes, i.e., where $k = 0, 1$, and at the mid-point of the horizontal segment. We denote the density at this mid-point $k = \hat{\boldsymbol{k}}_{\boldsymbol{o}}$ since it is the maximum of the estimated MFD. Consideration of the problem's geometry shows that:[1]

$$\hat{\boldsymbol{k}}_{\boldsymbol{o}} = 1/2 + (k_o - 1/2)\boldsymbol{q_o}. \tag{22}$$

Our choice for the MFD approximation is the following continuously differentiable power function, which has all the above-mentioned properties:

$$\text{For } k \leq \hat{\boldsymbol{k}}_{\boldsymbol{o}}: \qquad \boldsymbol{q} \approx \boldsymbol{q_o}\Big(1 - \Big(1 - \frac{k}{\hat{\boldsymbol{k}}_{\boldsymbol{o}}}\Big)^a\Big), \qquad \text{where } a = \hat{\boldsymbol{k}}_{\boldsymbol{o}}\boldsymbol{v_f}/\boldsymbol{q_o}. \tag{23a}$$

$$\text{For } k > \hat{\boldsymbol{k}}_{\boldsymbol{o}}: \qquad \boldsymbol{q} \approx \boldsymbol{q_o}\Big(1 - \Big(\frac{k - \hat{\boldsymbol{k}}_{\boldsymbol{o}}}{1 - \hat{\boldsymbol{k}}_{\boldsymbol{o}}}\Big)^b\Big), \qquad \text{where } b = (1 - \hat{\boldsymbol{k}}_{\boldsymbol{o}})\boldsymbol{w}/\boldsymbol{q_o}. \tag{23b}$$

In both (22) and (23) one can use (17) for $\boldsymbol{q_o}$. This is the recipe used for the dashed lines of Fig. 5.

5 Discussion

The results in this paper pertain to a homogeneous pedestrian flux. We expect this to be a best case scenario since as was pointed out at the outset of this paper, inhomogeneous

[1] Note the rays defining the two slanted sides of the trapezium intersect each other below the line $q = 1$, and then intersect this line at two equidistant points from the apex of the FD. Thus, the two rays and the three parallel lines $q = 1$, $q = \boldsymbol{q_o}$ and $q = 0$ define three similar triangles with parallel horizontal sides and a common vertex. As such, the triangles share a single median line that passes through their common vertex and bisects their horizontal sides. The condition that a single straight line passes through the three midpoints of the triangles' horizontal sides (i.e., points: $(k_o, 1)$, $(\hat{\boldsymbol{k}}_{\boldsymbol{o}}, \boldsymbol{q_o})$ and $(1/2, 0)$) reduces to (22).

flux distributions lead to higher delays for isolated cars and less throughput flow. Thus, research to explore the effects on inhomogeneous distributions is desirable. We anticipate that this research can benefit from the dimensional simplifications introduced in Sec. 2, and from VT.

Acknowledgement

Research supported by a research grant from UC-Connect and a research grant from the Netherlands Organisation of Scientific Research.

References

Daganzo, C.F. 1977, Traffic Delay at Unsignalized Intersections: Clarification of Some Issues, Transportation Science, 11, 180-189.

Daganzo, C.F. 1997, Fundamentals of Transportation and Traffic Operations, Pergamon-Elsevier, New York, N.Y.

Daganzo, C.F., 2005. A variational formulation of kinematic waves: basic theory and complex boundary conditions. Transportation Research Part B: Methodological, 39(2), 187-196.

Daganzo, C.F., 2005a. A variational formulation of kinematic waves: solution methods. Transportation Research Part B: Methodological, 39(10), 934-950.

Daganzo, C.F., 2006. On the variational theory of traffic flow: well-posedness, duality and applications. Networks and Heterogeneous Media, 1(4) 601-619.

Edie, L.C., 1963. Discussion of traffic stream measurements and definitions. Proc. 2nd International Symposium on the Theory of Traffic Flow, J. Almond, editor, pp. 139-154, OECD, Paris, France.

Feller, W., 1968. An Introduction to Probability Theory and its Applications, Vol I (3rd edition). John Wiley and Sons, New York, N.Y.

Hawkes, A.G., 1965. Queuing for gaps in traffic, Biometrika 52, 79-85.

Hawkes, A.G., 1968. Gap acceptance in road traffic, Journal of Applied Probability 5, 84-92.

Lighthill, M.J. and G.B. Whitham, 1955. On kinematic waves. I flow movement in long rivers. II A theory of traffic flow on long crowded roads. Proc. Roy. Soc. A 229, 281-345.

Newell, G.F., 1979. Approximate Behavior of Tandem Queues. Vol. 171 in Lecture Notes in Economics and Mathematical Sciences, Springer-Verlag, Berlin, Germany.

Newell, G.F., 1993. A simplified theory of kinematic waves: (i) general theory; (ii) queuing at freeway bottlenecks; (iii) multi-destination flows. Transportation Research Part B 37, 281-313.

Newell, G.F., 2002. A simplified car-following theory: a lower order model. Transportation Research Part B: methodological 36, 195-205.

Richards P.I., 1956. Shockwaves on the highwy. Opns. Res. 4, 42 51.