Kanazawa University founded in 1862

Kanazawa Flat Bus KUTE Kanazawa University Transportation Engineering lab.

q-Generalized logit route choice and network equilibrium model

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Abstract

The multinomial logit model is extended by generalizing the Gumbel-distributed utility to allow heteroscedastic variance and flexible shape. The generalized logit model with a generalized Gumbel distribution (generalized extreme value distribution) is incorporated into the transportation network equilibrium model.

error term	discrete choice model	(Boltzmann statistics)
Gumbel distribution ⇒	Logit model	⇐ Shannon entropy
↓ q-generalization	↓ q-generalization	↓ q-generalization
Generalized extreme distribution ⇒	**Generalized logit model**	⇐ Tsallis entropy (Tsallis statistics)

1. Generalization of error term in logit model

3 types of extreme value distribution

- Gumbel type → Gumbel distribution
- Frechet type
- Weibull type

⇩

GEV: Generalized extreme value distribution

Random utility of *i*-th alternative, U_i:

$$U_i = v_i + \varepsilon_i \qquad v_i: \text{deterministic utility}$$

Gumbel error term in MNL

↓ generalization

Generalized extreme value distribution

2. q-functions & generalized logit model

q-exponential function:

$$\exp_q(x) := [1+(1-q)x]^{\frac{1}{1-q}} \xrightarrow{q\to 1} \exp_1(x) = \exp(x)$$

$$\because \exp(x) = \lim_{\gamma\to 0}(1+\gamma x)^{\frac{1}{\gamma}}$$

q-logarithm function:

$$\ln_q(x) := \frac{x^{1-q}-1}{1-q} \xrightarrow{q\to 1} \ln_1(x) = \ln(x)$$

Cumulative distribution function of Generalized Extreme Value distribution:

$$\exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\theta}\right)\right]^{-\frac{1}{\gamma}}\right\} \Rightarrow G_i(x) = \exp\left[-\exp_q(\hat{v}_i)\exp_q\left(-\frac{x}{s}\right)\right]$$

Let $\gamma = q-1,\ \mu = v_i,\ \theta = s-(1-q)v_i$ $\qquad \hat{v}_i := \dfrac{v_i}{s-(1-q)v_i}$

Probability of choosing 1st alternative:

$$p_1 = \int_{x\in\Omega} g_1(x)G_2(x)\cdots G_I(x)\,dx = \frac{\exp_q(\hat{v}_1)}{\sum_{i=1}^{I}\exp_q(\hat{v}_i)}$$

$$g_1(x) = \frac{d}{dx}G_1(x)$$

Generalized logit

$$\boxed{p_i = \frac{\exp_q(\hat{v}_i)}{\sum_{i'=1}^{I}\exp_q(\hat{v}_{i'})}}$$

$s=1$, $q\to 1$ →; ← generalization

Standard logit

$$p_i = \frac{\exp(v_i)}{\sum_{i'=1}^{I}\exp(v_{i'})}$$

3. Tsallis entropy

$$S_q(\mathbf{p}) = -\frac{1-\sum_{i=1}^{I} p_i^{\,q}}{1-q} = -\sum_{i=1}^{I} p_i^{\,q}\ln_q(p_i)$$

$\downarrow q \to 1$

$$-\sum_{i=1}^{I} p_i \ln(p_i)$$

Shannon entropy (normal entropy)

4. Generalized Logit Assignment

$$f_{ij} = d_i \frac{\exp_q(-\theta c_{ij})}{\sum_{j'=1}^{J_i}\exp_q(-\theta c_{ij'})}$$

f_{ij}: flow on *j*-th route (*i*-th OD pair)
c_{ij}: travel time
d_i: demand (*i*-th OD pair)
θ: positive parameter

non-congested network

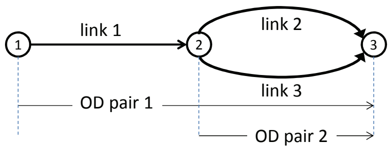
$$\min_{\mathbf{p}}.\ \sum_{i=1}^{I}\sum_{j=1}^{J_i} p_{ij}^{\,q} c_{ij} - \frac{1}{\theta}\sum_{i=1}^{I} S_q(\mathbf{p}_i)$$

Tsallis entropy

$$s.t.\ \sum_{i=1}^{I} p_i = 1 \qquad p_{ij} = \frac{f_{ij}}{d_i}$$

A relaxation algorithm can be applied to congested network cases

2-OD-3-link network example



$$t_1(x) = t_3(x) = 15\left[1+\left(\frac{x}{200}\right)^2\right]$$

$$t_2(x) = 10\left[1+\left(\frac{x}{100}\right)^2\right]$$

Set $q_1 = q_2 = q$, $s_1 = s_2 = s = 0.5$, and $\mu_{11} = \mu_{12} = \mu_{21} = \mu_{22} = 0$

Fig: Route choice probability over parameter q