Computational precision of traffic equilibria sensitivities in automatic network design and road pricing

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Overview

Introduction, motivation, and purpose

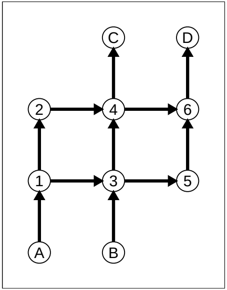
- Computational precision of traffic equilibria is important for comparing scenarios
- When utilized in a hierarchical model it is even *more* important, since we need derivative information
- For simplicity we study a fixed demand static model
- Contributions:
  1. A method for *precise* computations of derivatives
  2. Numerical investigations on the precision of derivatives, and its impact on network design solutions

Overview

Traffic assignment by paired alternative segments (TAPAS), I

- TAPAS (H. Bar-Gera, 2010) was designed for the purpose of solving the user equilibrium problem to a high precision (AEC $\leq 10^{-12}$), fast
- Storing a set of PASs means that costs can easily be equilibrated through shifting flows between segments
- The PASs define a compact representation of the routes utilized, and the flow circulation subspace
- The latter is useful when calculating directional derivatives of equilibrium flows, as its feasible set is a subset of this subspace
- Currently only the fixed-demand case is solved using TAPAS

Traffic assignment with pairwise alternative segments (TAPAS), II



Link	[A,1]	[B,3]	[1,3]	[3,5]	[1,2]	[3,4]	[5,6]	[2,4]	[4,6]	[4,C]	[6,D]
PAS_1	0	0	1	0	−1	1	0	−1	0	0	0
PAS_2	0	0	0	1	0	−1	1	0	−1	0	0
PAS_3	0	0	1	1	−1	0	1	−1	−1	0	0

Overview

Wardrop's conditions of user equilibrium

- (OD) pair (p, q) of nodes
- Finite set $\mathcal{R}_{pq}$ of simple (cycle-free) routes starting at node p and ending at node q
- h_r is the volume of traffic on route $r \in \mathcal{R}_{pq}$
- c_r is the travel cost on this route as experienced by an individual user, given the current volume h of traffic
- Wardrop:

$$\begin{aligned} h_r > 0 &\implies c_r = \pi_{pq}, \qquad r \in \mathcal{R}_{pq}, \qquad (p,q) \in \mathcal{C}, \\ h_r = 0 &\implies c_r \geq \pi_{pq}, \qquad r \in \mathcal{R}_{pq}, \qquad (p,q) \in \mathcal{C}, \end{aligned}$$

where π_{pq} denotes the minimal (that is, equilibrium) route cost for OD pair (p, q)

A variational inequality representation, I

- Standard assumption: route costs (c) equal sum of link costs (t)
- Consequential relations between flows and costs on routes and links:

$$v = \Lambda h; \qquad c(h) = \Lambda^{\mathrm{T}} t(v),$$

where $\Lambda \in \mathbb{R}^{|\mathcal{R}| \times |\mathcal{L}|}$ is the link–route incidence matrix

- Let

$$H := \{\, h \in \mathbb{R}_{+}^{|\mathcal{R}|} \mid \Gamma^{\mathrm{T}} h = d \,\}$$

denote the set of feasible route flows; then,

$$V := \left\{ v \in \mathbb{R}^{|\mathcal{L}|} \,\middle|\, \exists h \in H : \Lambda h = v \right\},$$

is the set of feasible link flows

- Wardrop's conditions as (equivalent) variational inequalities:

$$c(h^*)^{\mathrm{T}}(h - h^*) \geq 0, \qquad h \in H,$$

or,

$$t(v^*)^{\mathrm{T}}(v - v^*) \geq 0, \qquad v \in V$$

A variational inequality representation, II

- For what follows, a more natural (and compact) way to represent the above conditions is based on the normal cone operator, defined as

$$N_P(x) := \{\, p \in \mathbb{R}^n \mid p^{\mathrm{T}}(y - x) \leq 0, \qquad \forall y \in P \,\}$$

(here, P is a closed and convex set in $\mathbb{R}^n$)

- The VI "$f(x^*)^{\mathrm{T}}(x - x^*) \geq 0, \quad \forall x \in P$" is then the same as stating that

$$-f(x^*) \in N_P(x^*)$$

The design/pricing problem

- We introduce a set $X := \{\, x \in \mathbb{R}^{|\mathcal{L}|} \mid x^L \leq x \leq x^U \,\}$ of link "design" parameters—they represent actions in the network which influence the perception of travel time/cost
- We (compactly) represent the Wardrop conditions thus:

$$S(x) := \left\{ v \in \mathbb{R}^{|\mathcal{L}|} \,\middle|\, 0 \in t(x,v) + N_V(v) \right\}:$$

  at x the set $S(x)$ contains all user equilibrium link flows v
- For design/pricing purposes we wish to minimize a function F of x and the "response" $v = S(x)$; we write this problem as that to

$$\begin{aligned} &\underset{x}{\text{minimize}} \quad \hat{F}(x) := F(x, S(x)), \\ &\text{subject to} \quad x \in X \end{aligned} \tag{1}$$

- This is the "implicit programming" representation of an MPEC

Calculating sensitivities, I

- It is natural to attack (1) utilizing a descent method, based on the calculation of (sub)derivatives of $\hat{F}$
- Under some conditions on t directional derivatives of $\hat{F}$ exist; they are in fact found through the solution of a perturbed version of the original traffic equilibrium model:

$$-\nabla_x t(x^*, v^*)x' \in \nabla_v t(x^*, v^*)v' + N_K(v'), \tag{2}$$

where K [or, $K(v^*)$] is the critical cone of V at v^*, i.e.,

$$K = \left\{ v' \in \mathbb{R}^{|\mathcal{L}|} \;\middle|\; v' \in T_V(v^*) \text{ and } (v')^{\mathrm{T}} t(x^*, v^*) = 0 \right\},$$

and $T_V(v^*)$ is the tangent cone of V at v^*

- The cone K is composed by the link flow shifts v' for which perturbed link flows are first-order feasible and optimal with respect to the traffic equilibrium problem around v^*

Calculating sensitivities, II

- Solving (2) amounts to solving an affine VI, which in the strictly complementary case (i.e., all shortest routes are used) reduces to a linear system
- The utilization of TAPAS means that a compact (small) representation of K can be constructed from the PASs stored
- The linear system is solved using MINRES, which handles the fact that the system matrix may not have full rank
- A full "Jacobian" is constructed utilizing $|\mathcal{L}|$ unit perturbation vectors
- In practice degeneracy has not been a concern

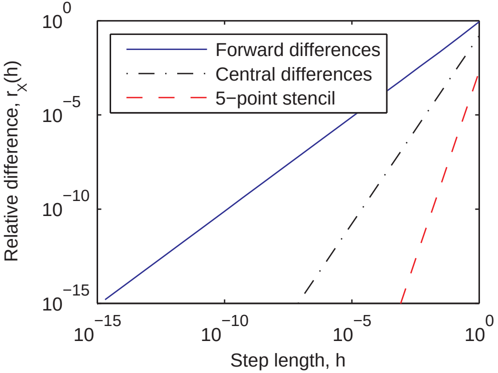
Overview

Overview

I: A general method for evaluating derivative precision

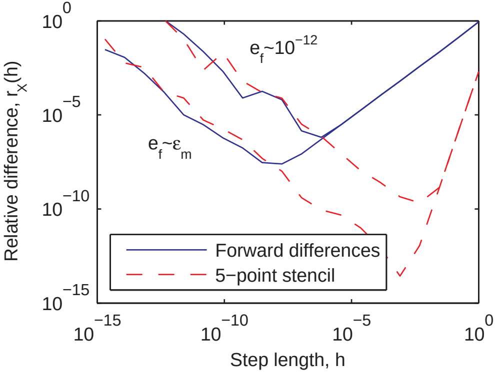
- We have implemented the sensitivity analysis-based derivative computation
- Interesting to compare its precision relative difference approximations of the derivative:
  forward-difference $\tilde{g}_F^{FD}(h;x) := \frac{\tilde{f}(x+h)-\tilde{f}(x)}{h}$
  central-difference $\tilde{g}_C^{FD}(h;x) := \frac{\tilde{f}(x+h)-\tilde{f}(x-h)}{2h}$
  five point stencil $\tilde{g}_5^{FD}(h;x) := \frac{-\tilde{f}(x+h)+8\tilde{f}\left(x+\frac{1}{2}h\right)-8\tilde{f}\left(x-\frac{1}{2}h\right)+\tilde{f}(x-h)}{6h}$
- Introducing numerical errors sources in the function value and/or in the derivative, we may study the effects on the derivative error. We chose $f(x) = \sin(x)$ at $x = 1$

No error in f or f'



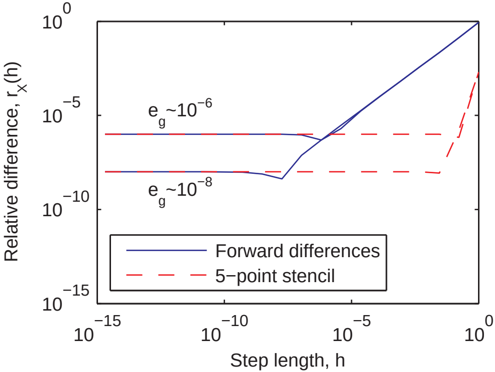
- Slopes correspond to the $O(h)$, $O(h^2)$ and $O(h^4)$ terms in the relative differences

Error in f only



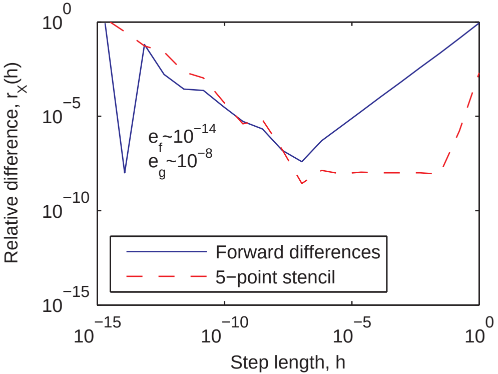
- Two plots: (a) all computations are made in 8-byte floating point arithmetic (error ε_m); (b) an additional relative error of 10^{-12} is added. We observe the $O(h^{-1})$ term dominating for small enough h

Error in f' only



- Two plots: one for a case where a relative error of 10^{-6} is added to f', and one case where a relative error of 10^{-8} is added. We can see the h-independent $e_{f'}$ term dominating for small enough h

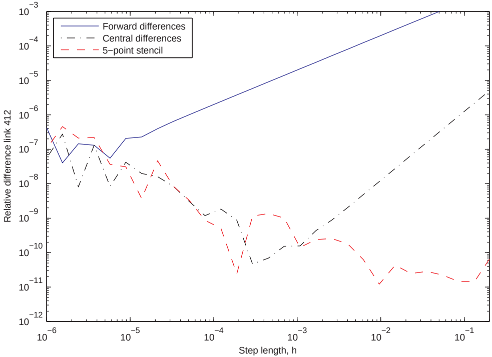
Error in f and f'



- Relative errors of 10^{-14} and 10^{-8} are added to f and f', respectively. We can see all three terms dominating for different values of h in the five point stencil graph (dashed line), while the forward difference scheme (solid line) is too imprecise to render the derivative error

II: Evaluating gradient precision in capacity expansion

- The Anaheim network (1406 OD-pairs, 416 nodes, 914 links) was used to investigate the difference between the "exact" derivative and the use of finite difference schemes; link 412 chosen



- Neither forward or central difference were able to reach the same accuracy as the sensitivity analysis computation

Overview

III: Network design

- We investigated the effect of precise derivative computations on the ability to solve MPECs (or, bilevel programs)

- We took the well-known Sioux Falls capacity expansion network where ten links can be invested in; the objective is to minimize the sum of total travel cost and investment cost

- We used two general solvers (SNOPT & PBUN) where the objective function and derivatives were calculated using our approach, and compared with three previous attempts on solving the problem: CSP (Chiou, 2008), PIPA (Lim, 2002), SBD (Josefsson & Patriksson, 2007)—from two initial guesses

II: Sioux Falls—initial guess 0

Method	Proposed method SNOPT	Proposed method PBUN	PIPA	SBD
x_{16}	5.2487	5.3419	5.4680	5.3027
x_{17}	1.4993	2.0711	2.0039	2.0560
x_{19}	5.2768	5.3688	5.4471	5.3430
x_{20}	1.4755	2.0438	1.9395	1.9901
x_{25}	2.9207	2.5223	2.9448	2.5216
x_{26}	2.9681	2.5672	2.8191	2.5548
x_{29}	3.9282	2.7693	3.4039	2.9883
x_{39}	4.8922	4.7923	4.8061	4.8559
x_{48}	3.9622	2.8258	3.2364	3.0026
x_{74}	4.8819	4.7798	4.7779	4.8496
F (reported)	80.1461	79.9003	80.8669	79.9961
F (recalc.)	80.1461	79.9003	79.9601	79.9126
Eq. sol. count	72	55	?	?
Time (s.)	12.4	8.9	?	?

II: Sioux Falls—initial guess 6.25

Method	Proposed method			
	SNOPT	PBUN	CSP	SBD
x_{16}	5.3520	5.3275	4.1007	5.2773
x_{17}	2.0732	2.0629	4.1254	2.0533
x_{19}	5.3813	5.3667	1.9807	5.3002
x_{20}	2.0455	2.0294	1.4532	2.0369
x_{25}	2.4603	2.5174	1.5643	2.7670
x_{26}	2.5102	2.5493	3.0987	2.8222
x_{29}	2.7715	2.7673	4.9876	3.0124
x_{39}	4.8032	4.8057	5.1095	4.7348
x_{48}	2.8279	2.8256	4.7896	2.9746
x_{74}	4.7888	4.7728	4.1287	4.7511
F (reported)	79.8999	79.9003	81.04	80.0043
F (recalc.)	79.8999	79.9003	82.2552	79.9237
Eq. sol. count	512	30	?	?
Time (s.)	85.9	5.3	35 (in 2005)	?

Remarks and future research

- It is interesting to note the big differences in the number of equilibrium computations needed: for SNOPT, 72 for the starting point 0, and 512 for the starting point 6.25. Further, in the former case SNOPT reports that optimality conditions are satisfied, while in the latter case it terminated because of numerical difficulties
- An additional investigation showed that the former point is differentiable, while the latter is not
- Profiling: for PBUN, out of 86 CPU s., TAP utilized 56 s.; the sensitivity analysis & MINRES 13 s.; and the remaining 17 s. on evaluating functions at the upper level
- In the future we intend to extend the analysis to the elastic demand case; it is straightforward in theory