Macroscopic Modeling Framework Unifying Kinematic Wave Modeling and Three-Phase Traffic Theory

Serge P. Hoogendoorn, Hans van Lint, and Victor L. Knoop

Modeling breakdown probabilities or phase-transition probabilities is an important issue when assessing and predicting the reliability of traffic flow operations. Looking at empirical spatiotemporal patterns, these probabilities clearly are a function not only of the local prevailing traffic conditions (density, speed) but also of time and space. For instance, the probability that a start–stop wave occurs generally increases when moving upstream away from the bottleneck location. A simple partial differential equation is presented that can be used to model the dynamics of breakdown probabilities, in conjunction with the well-known kinematic wave model. The main assumption is that the breakdown probability dynamics satisfy the way information propagates in a traffic flow, that is, they move along with the characteristics. The main result is that the main characteristics of the breakdown probabilities can be reproduced. This is illustrated through two examples: free flow to synchronized flow (F-S transition) and synchronized to jam (S-J transition). It is shown that the probability of an F-S transition increases away from the on ramp in the direction of the flow; the probability of an S-J transition increases as one moves upstream in the synchronized flow area.

The research of Kerner has resulted in quite a stir in the traffic flow theory community (*1*). Among the issues raised by Kerner are that there are three phases (free flow, synchronized flow, and jams), rather than two (free flow and congestion); that the breakdown phenomenon is a stochastic process stemming from the fact that small or large disturbances can trigger phase transitions with a certain probability; and that the fundamental diagram does not exist since the congested branch is a two-dimensional area, rather than a straight line. Furthermore, Kerner claims that none of the current microscopic or macroscopic traffic flow models captures correctly the different flow characteristics that are observed from empirical analyses.

This paper focuses on the breakdown phenomenon. A unifying modeling framework is proposed that allows the dynamics of the breakdown or phase-transition probabilities to be modeled intuitively by using the kinematic wave model (*2, 3*) as a basis. Various researchers have considered the dynamic modeling of breakdown probabilities (*4–7*), commonly using (stochastic) queuing analysis and nucleation models. These modeling approaches are thoroughly discussed elsewhere (*7*).

Here, use is proposed of a coupled set of partial differential equations describing both the traffic dynamics [using the first-order model of Lighthill and Witham (*2*) and Richards (*3*)] and the dynamics of the phase-transition probabilities. The proposed modeling framework can be considered as a relatively straightforward generalization of the kinematic wave theory to three-phase theory.

The focus is on dynamic modeling of the phase-transition probabilities and the implications this has for the properties of the first-order model. Other issues discussed by Kerner (such as the two-dimensional area depicting stable states in synchronized flow) are not considered in this paper.

Transport and Planning Department, Faculty of Civil Engineering and Geosciences, Delft University of Technology, Stevinweg 1, P.O. Box 5048, 2600 GA Delft, Netherlands. Corresponding author: S. P. Hoogendoorn, s.p.hoogendoorn@tudelft.nl.

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MATHEMATICAL MODEL OF BREAKDOWN PROBABILITY

This contribution describes dynamic modeling of the breakdown (or, rather, phase transition) probability, which is denoted by $P = P(t, x)$. The probability is a function of time t and space x and is thus not only determined by the prevailing traffic conditions such as the density. The macroscopic dynamic model consists of the following set of coupled ordinary differential equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

where $c(\rho) = \dfrac{dQ}{d\rho}$

$$\frac{\partial P}{\partial t} + c(\rho)\frac{\partial P}{\partial x} = \pi(\rho, P) \tag{2}$$

with initial conditions

$$\begin{cases} \rho(0, x) = \rho_0(x) \\ P(0, x) = P_0(x) \end{cases} \qquad 0 \le x \le L \tag{3}$$

where L denotes the roadway length. The boundary conditions are usually defined at the road entry, by specification of the inflow $q_0(t)$.

In this case, the transition probability at the boundary is to be defined:

$$\begin{cases} q(t,0) = q_0(t) \\ P(t,0) = p_0(t) \end{cases} \quad t \geq t_0 \tag{4}$$

Equation 4 implies that all vehicles can freely enter the road, although in practical situations, this is not necessarily the case (e.g., in case of congestion spillback). For the boundary conditions at $x = L$, it is assumed that all traffic can exit the road freely, so no boundary conditions are given for the exit.

In Equations 1 and 2, $c(\rho)$ denotes the kinematic wave speed, describing the speed (and the direction!) at which (small) perturbations propagate through the traffic flow. The kinematic wave speed is equal to the derivative of the fundamental diagram $Q = Q(\rho)$. This follows directly from the shock-wave equation, stating that the speed of a shock wave S separating regions (ρ_1, q_1) and (ρ_2, q_2) is given by

$$\omega = \frac{q_2 - q_1}{\rho_2 - \rho_1} = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1} \rightarrow \lim_{\rho_2 \to \rho_1} \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1} = \frac{dQ}{d\rho}(\rho_1) = c(\rho_1) \tag{5}$$

In Equation 2, $\pi = \pi(\rho, P)$ denotes the rate of change in the breakdown probabilities P, which are assumed to be a function of the density $\rho = \rho(t, x)$ and the probability P itself. Note that P can describe both a free-flow to synchronized flow (F-S) transition ($P = P_{\text{F-S}}$) or a synchronized to jam (S-J) transitions ($P = P_{\text{S-J}}$). Both examples are shown.

Although P can be construed as a probability in the classical sense of the word, for its corrective interpretation one also needs to consider the time and space dimensions explicitly. That is, P denotes the probability that a phase transition will occur within the next τ seconds somewhere in a roadway segment of length d. In numerical studies, τ will be generally chosen equal to the simulation time step Δt, d will be chosen equal to the cell length Δx, and P will denote the probability that a phase transition occurs during this time step.

Model Justification

The concept behind the mathematical model is the assumption that the phase-transition probability P changes along the characteristic curves, just as the density does. This means that if a perturbation in the flow is considered, the phase-transition probability P will change along with this perturbation moving with the kinematic wave speed $c(\rho)$.

To understand this property fully, consider a platoon of vehicles. Suppose that the platoon leader will brake briefly. The disturbance this causes will move from one vehicle to the next, possibly changing in amplitude while moving upstream. The speed at which this perturbation moves is equal to the characteristic speed $c(\rho)$. If the perturbation becomes sufficiently large, it may induce a phase transition, depending on the stability conditions given by the prevailing traffic conditions. Alternatively, the perturbation may damp out, implying that the probability of a phase transition will reduce along the perturbation.

Consider the so-called characteristic curves of the kinematic wave model. These curves are parameterized curves C that are defined by the path

$$C = \{t(s), x(s)\} \tag{6}$$

In Equation 6, $t = t(s)$ and $x = x(s)$ are defined by the following ordinary differential equations:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = 1$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}s} = c(\rho) \tag{7}$$

Now, let $\rho(s) = \rho(t(s), x(s))$ denote the (parameterized) density along the characteristic curve:

$$\frac{\mathrm{d}\rho}{\mathrm{d}s} = \frac{\mathrm{d}\rho}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\mathrm{d}\rho}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{8}$$

The density ρ is conserved along the characteristic C ($\rho(s) = \rho(0)$). Since the characteristic speed $c(\rho)$ depends on the constant density ρ only, the speed $c(\rho)$ is constant as well. Therefore, the characteristic curves C in the kinematic wave model are straight lines.

The same characteristic curves can be used for Equation 2. Let $P = P(s)$ denote the phase-transition probability along C. One can thus show that

$$\frac{\mathrm{d}P}{\mathrm{d}s} = \frac{\mathrm{d}P}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\mathrm{d}P}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial P}{\partial t} + c(\rho)\frac{\partial P}{\partial x} = \pi(\rho, P) \tag{9}$$

Since $\mathrm{d}P/\mathrm{d}s = \pi(\rho, P)$, $\pi(\rho, P)$ can be interpreted as the rate at which the phase transition probability changes over time along the characteristic.

Inside a congested region, $c(\rho) \approx -15$ km/h ≈ -4 m/s, implying that the phase-transition probability P increases in the move upstream away from the point at which the congestion originated. Considering $P = P_{\text{S-J}}$ (transition from synchronized to jammed flow), it can be modeled that the probability of a transition from synchronized flow to so-called wide-moving jams increases when moving away from the head of the queue (in the upstream direction).

Outside congestion, one has $c(\rho) \approx 85$ km/h. Considering $P = P_{\text{F-S}}$ (probability of a transition from free flow to synchronized flow), the observed increases in this probability can be modeled moving downstream from the bottleneck, for example, that congestion sets in downstream of an on ramp rather than at the location of the on ramp itself.

Behavior of Phase Transition Probabilities at Shocks

Shock waves occur when characteristics C intersect. If this happens, the speed ω of the shock is determined by the well-known shock wave equations in Equation 5 and is determined only by the flows and the densities upstream and downstream of the shock. The phase-transition probabilities upstream and downstream of the shock are not affected by the shock itself, because the characteristics move toward the shock (and do not emanate from the shock).

Discretization of Equations

To numerically solve the problem, the standard Godunov scheme is proposed for the conservation of vehicle equation (*8*). For the transition probability, basically any discretization scheme will work. The following standard scheme is proposed:

$$P_{i,j+1} = P_{i,j} - \Delta t \cdot \left(c^{+}\left(\rho_{i,j}\right) \cdot \frac{P_{i+1,j} - P_{i,j}}{\Delta x} + c^{-}\left(\rho_{i,j}\right) \cdot \frac{P_{i,j} - P_{i-1,j}}{\Delta x} \right) + \pi\left(\rho_{i,j}, P_{i,j}\right) \quad (10)$$

where

$$c^{+}\left(\rho_{i,j}\right) = \max\left(0, c\left(\rho_{i,j}\right)\right) \qquad c^{-}\left(\rho_{i,j}\right) = \min\left(0, c\left(\rho_{i,j}\right)\right) \quad (11)$$

SPECIFICATION OF MODEL RELATIONS

The main goal for this paper is to illustrate how the dynamics of phase transitions can be modeled by using the presented approach. The examples in the next sections show that the modeling outcomes resemble real-life flow patterns.

Fundamental Diagram and Phase Definitions

To illustrate the consequences of the dynamic modeling of phase transitions probabilities, a simple, piecewise linear fundamental diagram is used (Figure 1). The diagram has four parameters: the free capacity C_{free}, the queue discharge rate C_{cong}, the critical density ρ_{crit}, and the jam density ρ_{jam}.

The three phases considered in this contribution are free-flow conditions F (the left branch of the diagram), the synchronized flow conditions S (the right branch of the diagram), and jam conditions J. (For simplicity, a linear relation is assumed between density and flow in case of synchronized flow. Although this is not in line with three-phase theory, it is sufficient in showing the model behavior.)

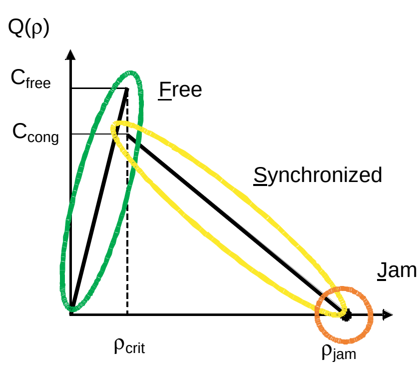

FIGURE 1 Piecewise linear fundamental diagram, including phase definitions.

Transitions from the one phase to the other are modeled by instantaneous jumps in the phase diagram. In the application example, these are instigated on the basis of the value of the phase-transition probabilities $P_{F\text{-}S}$ (free flow to synchronized flow) and $P_{S\text{-}J}$ (synchronized flow to jam). The different transitions are modeled as follows:

- A breakdown from F to S is modeled by a jump in the capacity (from C_{free} to C_{cong}), keeping density constant and equal to the critical density. In the example, it is assumed that this transition occurs whenever (and lasting as long as) $P_{F\text{-}S} \geq 0.5$.
- A transition from S to F is modeled by a jump in the capacity (from C_{cong} to C_{free}). For the example shown, this occurs whenever the $P_{F\text{-}S}$ drops below 0.5.
- A transition from S to J is modeled by assuming zero flow at the breakdown location (effectively reducing the local capacity to zero for a specific period). This state is maintained until a transition from J to S occurs. In the example presented, this occurs as long as (and at all locations where) $P_{S\text{-}J} \geq 0.5$.
- A transition from J to S is modeled by assuming maximum outflow from the jam. That is, the flow is restored from 0 to C_{cong}. In the example shown in the remainder, this occurs when $P_{S\text{-}J} < 0.5$.

This is only an example to illustrate the model properties. More refined approaches, such as using random transitions based on the phase transition probabilities, can be easily developed based on the ideas put forward in this paper.

Specification of Transition Probability Rate

Consider the phase-transition probability rates $\pi = \pi(\rho, P)$ and its relation to the phase-transition probabilities. A mathematical specification of these rates would consider the location where transitions from one traffic phase to the other traffic phase occurs, such that the (locations of) the transitions can be correctly on a phenomenological level.

The following linear expression is used for the rate $\pi(\rho, P)$ (both for the F-S transitions and the S-J transitions, be it with different parameter values; see Equation 12):

$$\pi(\rho, P) = \begin{cases} \left(\pi_0 + \pi_1 P\right) \cdot \dfrac{\rho - \rho_0}{\rho_1 - \rho_0} & \rho_0 \leq \rho \leq \rho_1 \\ 0 & \text{elsewhere} \end{cases} \quad (12)$$

Additionally, it is assumed that $P = 0$ if the density is less than ρ_0. Furthermore, P will be limited to values between 0 and 1. After some straightforward computations, it follows that along the characteristic curves $C = \{t(s), x(s)\}$, the transition probability equals

$$\mathrm{P}(s) = \min\left\{ \frac{\pi_0}{\pi_1}\left(e^{\pi_1\left(\frac{\rho-\rho_0}{\rho_1-\rho_0}\right)s} - 1 \right), 1 \right\} \quad (13)$$

EXAMPLE APPLICATION OF THEORY

Results are given in this section of applying the model. Both F-S transitions ($P = P_{F\text{-}S}$) and S-J transitions ($P = P_{S\text{-}J}$) are considered. First, the specification of the transition probability rates π used in the rest of the paper is given.

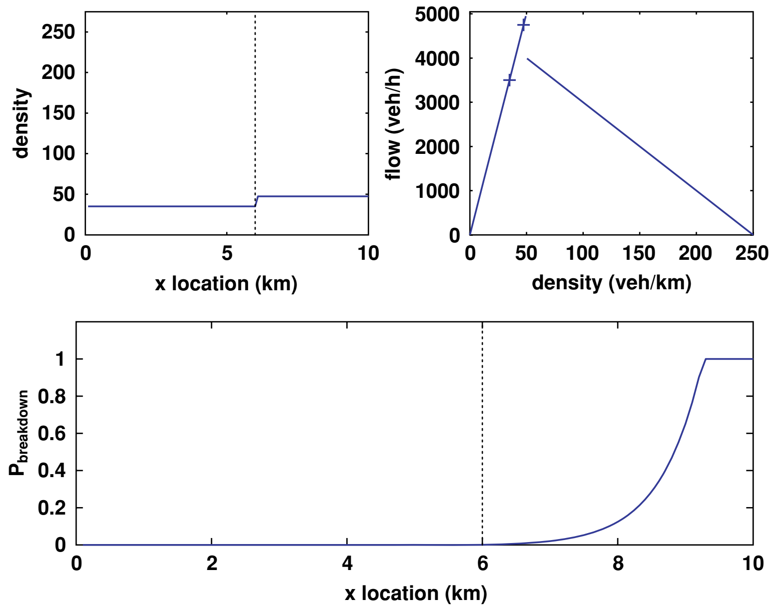

FIGURE 2 On-ramp scenario showing increase in F-S transition probability.

F-S Transition Probability Behavior

Consider a two-lane, 10-km road with an on ramp at $x = 6$ km. For the piecewise linear fundamental diagram, assume $C_{free} = 4{,}500$ veh/h, $C_{queue} = 4{,}000$ veh/h, $\rho_{crit} = 50$ veh/km, and $\rho_{jam} = 250$ veh/km.

For the scenario at hand, $Q_{main} = 3{,}500$ veh/h and $Q_{on\text{-}ramp} = 1{,}250$ veh/h. Various parameter values were considered and $\pi_0 = 1$ and $\pi_1 = 100$ were chosen (for illustration purposes); $\rho_0 = 40$ veh/km and $\rho_1 = \rho_{crit}$. It is assumed for all examples that all traffic coming from the on ramp will be able to merge onto the freeway.

Figure 2 shows the results of the numerical experiment. The figure shows the density profile, the location of the points on the fundamental diagram, and the transition probabilities. The F-S breakdown probability increases nonlinearly after the on ramp at $x = 6$ km. In other words, the occurrence of a breakdown becomes more likely further downstream of the on ramp. (Note that $\pi_1 = 0$ would yield a linear increase.)

If it is assumed there would be an F-S transition (in this case, modeled by temporarily assuming that the capacity is reduced from C_{free} to C_{queue}) when $P_{F\text{-}S} > 0.5$, the simulation shows that at a certain time instant the transition occurs (downstream of the bottleneck), moves upstream, and passes the on-ramp location. There it leads to the onset of congestion (because the capacity is reduced); see Figure 3. Figure 4 shows the resulting dynamics of the breakdown probability $P_{F\text{-}S}$ itself.

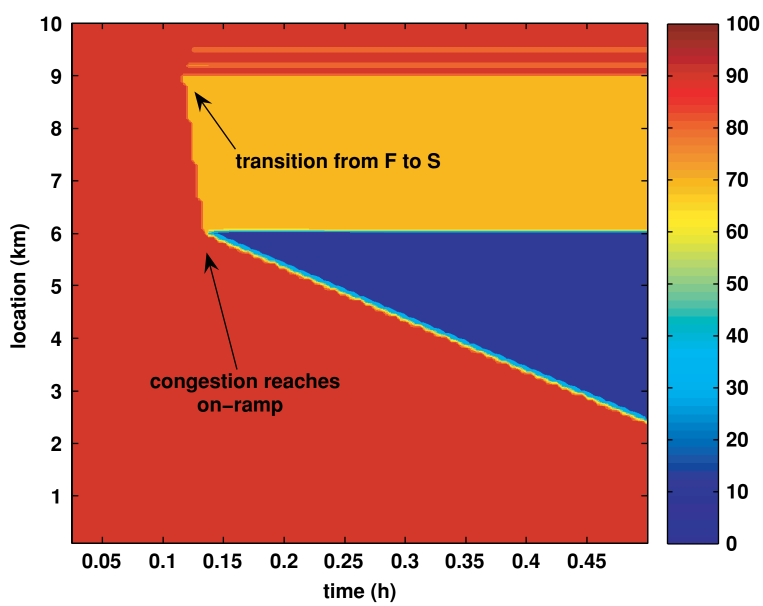

FIGURE 3 F-S transition showing how congestion sets in downstream of on-ramp and moves upstream.

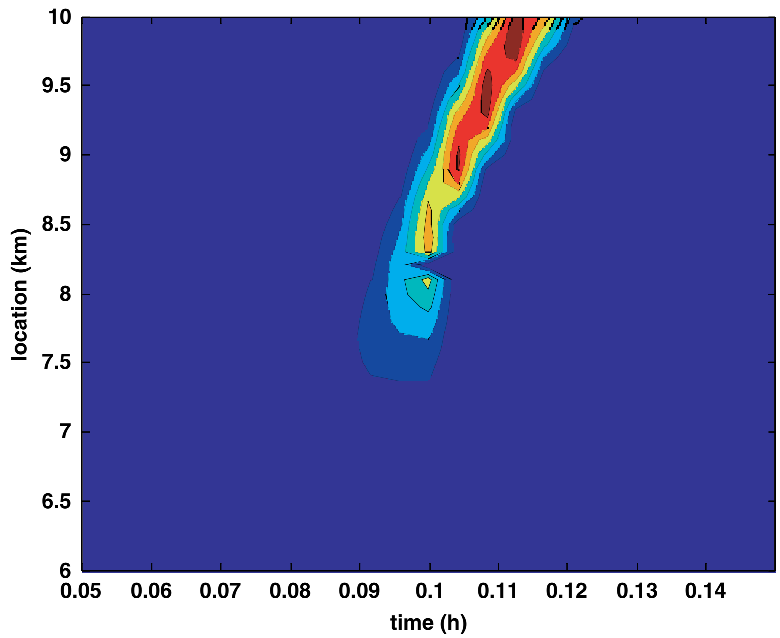

FIGURE 4 Dynamics of $P_{F\text{-}S}$ for on-ramp scenario.

S-J Transition Probability Behavior

Similar behavior is found for the S-J transitions. In this case, $\pi_0 = 1$ and $\pi_1 = 100$ (for illustration purposes) have again been used to describe the S-J transition; $\rho_0 = \rho_{crit}$ and $\rho_1 = 200$ veh/km. For this scenario, assume that $Q_{main} = 4{,}000$ veh/h and $Q_{on\text{-}ramp} = 1{,}500$ veh/h, implying that the bottleneck is oversaturated.

The result is indeed similar to the result found for the F-S transition: the probability on an S-J transition is zero at the on ramp (where this model assumes that the head of the queue is located) and increases nonlinearly in the move upstream away from the bottleneck (Figure 5).

As a final example, assume that an S-J transition occurs when $P_{S\text{-}J} > 0.5$ (i.e., it is deterministic). Figures 6 and 7 show the results of this analysis. Clearly, the values are not realistic, but the general picture appears to be correct. It is also astonishing to see the chaotic patterns that emerge even when this simple example is used.

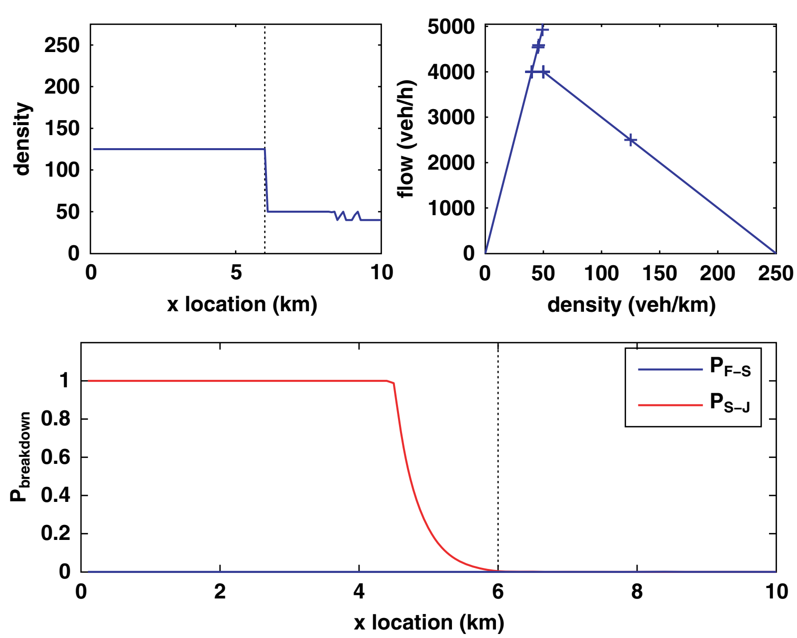

FIGURE 5 Probability of S-J breakdown, increasing in move from on ramp in upstream direction.

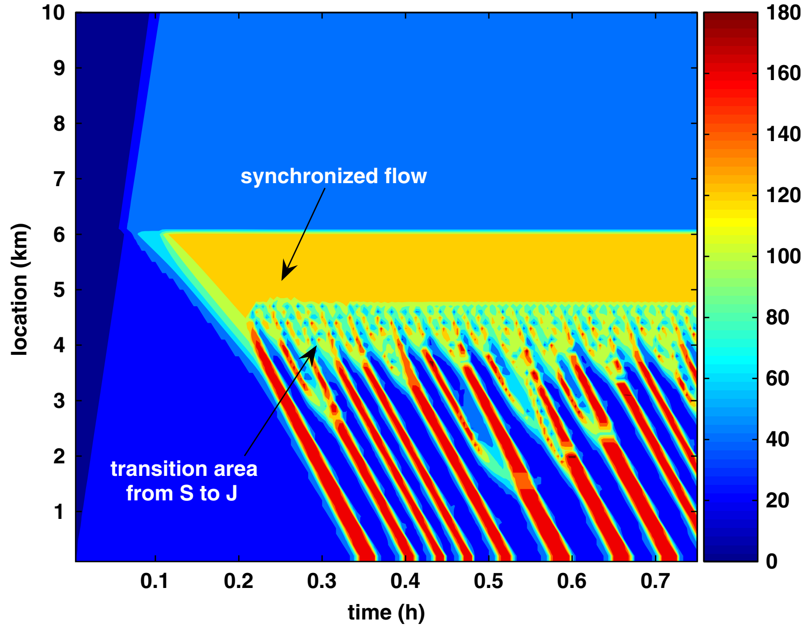

FIGURE 6 Deterministic modeling of S-J transition.

TOWARD STOCHASTIC MODELING

So far, there are only deterministic phase transitions where it is assumed that a phase transition (from F to S or from S to J) occurs deterministically when $P > 0.5$. This section discusses how the model can be extended toward a stochastic model. This will be performed for the discretized model, where it is assumed that the roadway was divided into cells of length Δx and the time step is equal to Δt.

It is assumed that at any time t, the time to breakdown (or rather, time to phase transition) within a cell of a fixed length is exponentially distributed with a mean breakdown time $\tau = \tau(t, x)$. That is, the probability that the random breakdown time T is less that s equals

$$P = \Pr(T < s) = 1 - \exp(-s/\tau) \quad (14)$$

For short intervals (e.g., of length $h > 0$), the probability that traffic breakdown will occur is equal to

$$P = \Pr(T < h) = 1 - \exp\frac{-\Delta t}{\tau} \approx \frac{\Delta t}{\tau} \quad (15)$$

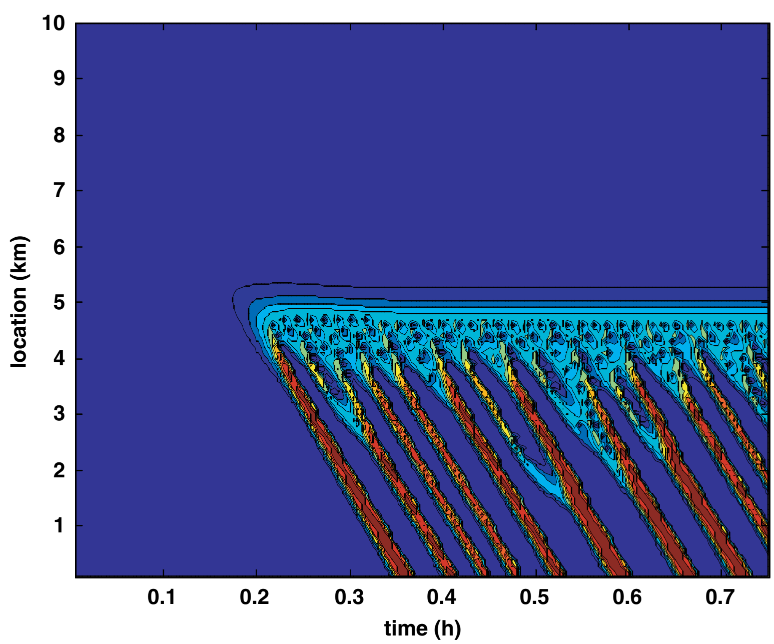

FIGURE 7 Dynamics of $P_{S\text{-}J}$ for on-ramp example.

For small time intervals, one can thus conclude that the breakdown probability in an interval of length $2\Delta t$ is twice as big as the breakdown probability in an interval of length Δt. This is intuitively correct.

The same line of thought is applied to the length of the cell being considered: intuitively, a cell of length Δx has twice the probability that traffic will break down than does a cell of length Δx. This is why Equation 14 is replaced and it is assumed that the breakdown probability P can be modeled via the following exponential distribution:

$$P = \Pr(T < \Delta t) = 1 - \exp\left(-\frac{\Delta t \cdot \Delta x}{\tau \cdot d}\right) \quad (16)$$

In Equation 16, τ is the mean time to breakdown determined for a roadway segment of length d.

CONCLUSIONS

This paper proposed a relatively simple extension of the first-order model pertaining to the inclusion of breakdown probabilities. The breakdown probability is modeled by using a partial differential equation. The main assumption is that information regarding the breakdown probability moves along the characteristic curves.

The workings of the model are illustrated by means of an example featuring flow breakdown due to an on ramp. By using this example, it is shown that the model can capture the main features of the different phase transitions (F-S, S-J).

Future research will model the phase transition itself. In the examples provided in this paper, this was achieved by using a simple threshold value for the transition probabilities. A stochastic approach, however, is more realistic. Clearly, this would yield a stochastic first-order macroscopic model.

Another extension of the theory is to use a multiclass traffic flow model, distinguishing between person cars and trucks. In doing so, the dynamics of the phase transitions can be made dependent on the traffic composition, since clearly this has a strong effect on the breakdown probability dynamics.

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