Dynamic First-Order Modeling of Phase-Transition Probabilities

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Summary. Modeling breakdown probabilities or phase transition probabilities is an important issue when assessing and predicting the reliability of traffic flow operations. Looking at empirical spatio-temporal patterns, these probabilities clearly are not only a function of the local prevailing traffic conditions (density, speed), but also of time and space. For instance, the probability that start-stop wave occurs generally increases when moving upstream away from the bottleneck location.

The dynamics of the breakdown probabilities are the topic of this paper. We propose a simple partial differential equation that can be used to model the dynamics of breakdown probabilities, in conjunction with a first-order model. The main assumption is that the breakdown probability dynamics satisfy the way information propagates in a traffic flow, i.e. they move along with the characteristics.

The main result is that we can reproduce the main characteristics of the breakdown probabilities, such as observed by Kerner. This is illustrated by means of two examples: free flow to synchronized flow (F-S transition) and synchronized to jam (S-J transition). We show that the probability of an F-S transition increases away from the on-ramp in the direction of the flow; the probability of an S-J transition increases as we move upstream in the synchronized flow area. Note that all the examples shown in the paper are deterministic.

1 Introduction

The research and claims of Kerner [1] has resulted in quite a stir in the traffic flow theory community. Amongst the issues raised by Kerner are the fact that there are three phases (free flow, synchronized flow and jams), rather than two (free flow and congestion), the fact that the breakdown phenomenon is a stochastic process stemming from the fact that small or large disturbances can trigger phase transitions with a certain probability, and the fact that the fundamental diagram does not exist since the congested branch is a 2D area, rather than a straight line.

Furthermore, Kerner claims that none of the current microscopic or macroscopic traffic flow models captures correctly the different flow characteristics that are observed from empirical analyzes.

This paper focuses on the breakdown phenomenon. More specially, the main contribution of the paper is that we show a first-order macroscopic modeling framework that allows us to model the dynamics of the breakdown or phase-transition probabilities in an intuitive and simple manner.

Different researchers have considered the dynamic modeling of breakdown probabilities (see [2–4]), commonly using (stochastic) queuing analysis. In this contribution, we propose using coupled set of partial differential equations describing both the traffic dynamics (using a simple first-order model) and the dynamics of the phase-transition probabilities. In other words, the proposed modeling framework can be considered as a relatively straightforward generalization of the kinematic wave theory.

Note that we focus on the dynamic modeling of the phase-transition probabilities, and the implications this has for the properties of the first-order model. Other issues discussed by Kerner (such as the 2D area depicting stable states in synchronized flow) are not considered.

2 Mathematical Model of Breakdown Probability

This contribution describes dynamic modeling of the breakdown (or rather phase transition) probability, which is denoted by $P = P(t, x)$. Note that the probability is a function of time t and space x, and is thus not only determined by the prevailing traffic conditions such as the density.

The macroscopic dynamic model consists of the following set of equations:

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

$$\frac{\partial P}{\partial t} + c(\rho)\frac{\partial P}{\partial x} = \pi(\rho, P) \tag{2}$$

In Eqs. (1) and (2), $c(\rho) = \frac{dQ}{d\rho}$ denotes the kinematic wave speed, describing the speed (and direction!) at which (small) perturbations propagate through the traffic flow. The kinematic wave speed is equal to the derivative of the fundamental diagram $Q = Q(\rho)$. This follows directly from the shockwave equation, stating that the speed of a shockwave S separating regions (ρ_1, q_1) and (ρ_2, q_2) is given by:

$$\omega = \frac{q_2 - q_1}{\rho_2 - \rho_1} = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1} \tag{3}$$

yielding:

$$\lim_{\rho_2 \to \rho_1} \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1} = \frac{dQ}{d\rho}(\rho_1) \tag{4}$$

In Eq. (2), $\pi = \pi(\rho, P)$ denotes the rate of change in the breakdown probabilities P, which are assumed to be a function of the density $\rho = \rho(t, x)$ and the probability P itself. Also note that P can describe both an F-S transition ($P = P_{F-S}$) or a S-J transitions ($P = P_{S-J}$). Both examples will be shown in the ensuing of the contribution.

2.1 Model Justification

The concept behind the mathematical model is the assumption that the phase-transition probability P changes along the *characteristic curves* (just as the density). This means that if we consider a perturbation in the flow, the phase-transition probability P will change along with this perturbation.

To understand this property, let us consider a platoon of vehicles. Suppose that the platoon leader will brake briefly. This perturbation will move from the one vehicle to the next, possibly changing in amplitude while moving upstream. The speed at which the perturbation moves is equal to the characteristic speed. If the perturbation becomes sufficiently large, it may induce a phase transition. Alternatively, the perturbation may damp out implying that the probability of a phase transition will reduce along the perturbation.

Let us now take a closer look at the characteristic curves. These curves are parameterized curves C that are defined by the path:

$$C = \{t(s), x(s)\} \tag{5}$$

where $t = t(s)$ and $x = x(s)$ are defined by the following differential equations:

$$\frac{dt}{ds} = 1 \text{ and } \frac{dx}{ds} = c(\rho) \tag{6}$$

Now, let $\rho(s) = \rho(t(s), x(s))$ denote the (parameterized) density along the characteristic curve. We have:

$$\frac{d\rho}{ds} = \frac{d\rho}{dt}\frac{dt}{ds} + \frac{d\rho}{dx}\frac{dx}{ds} = \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{7}$$

We thus see that the density ρ is conserved along the characteristic C (i.e. $\rho(s) = \rho(0)$). Since the characteristic speed $c(\rho)$ depends on ρ, the speed is constant as well, and thus the characteristic C is a straight line.

For (2), we can use the same characteristic curves. Let $P = P(s)$ denote the breakdown probability along C. We can thus show that:

$$\frac{dP}{ds} = \frac{dP}{dt}\frac{dt}{ds} + \frac{dP}{dx}\frac{dx}{ds} = \frac{\partial P}{\partial t} + c(\rho)\frac{\partial P}{\partial x} = \pi(\rho, P) \tag{8}$$

Since $dP/ds = \pi(\rho, P)$, $\pi(\rho, P)$ can be interpreted as the rate at which the breakdown probability changes over time along the characteristic.

Please note inside a congested region, $c(\rho) \approx -15$ km/h, implying that the breakdown probability increases as we move upstream away from the point at which the congestion originated. If we consider $P = P_{S-J}$ (transition from synchronized to free flow), we can thus model the fact that the probability of a transition form synchronized flow to wide moving jams increases when moving away from the head of the queue (in the upstream direction).

Outside congestion, we have $c(\rho) \approx 85$ km/h. If we now consider $P = P_{F-S}$ (probability that we have a transition form free flow to synchronized flow), we can thus model the observed increases in this probability as we proceed downstream from the bottleneck, e.g. the fact that congestion sets in downstream of an on-ramp rather than at the location of the on-ramp itself.

2.2 Discretization of the Equations

To numerically solve the problem, we propose using the standard Godunov scheme for the conservation of vehicle equation [5]. For the transition probability, basically any discretization scheme will work. We propose the following standard scheme:

$$P_{i,j+1} = P_{i,j} + \Delta t \cdot \pi(\rho_{i,j}, P_{i,j}) - \Delta t \cdot \left(c^{+}(\rho_{i,j}) \cdot \frac{P_{i+1,j} - P_{i,j}}{\Delta x} + c^{-}(\rho_{i,j}) \cdot \frac{P_{i,j} - P_{i-1,j}}{\Delta x} \right) \tag{9}$$

where

$$c^{+}(\rho_{i,j}) = \max(0, c(\rho_{i,j})) \text{ and } c^{-}(\rho_{i,j}) = \min(0, c(\rho_{i,j})) \tag{10}$$

3 Example Application of Theory

Let us now show some results of applying the model. In this section, we will consider both the F-S transitions ($P = P_{F-S}$) and the S-J transitions ($P = P_{S-J}$). Before showing these examples, we will present the specification of the transition probability rates π used in the remainder of the contribution.

3.1 Specification of the Fundamental Diagram

In this contribution, we use a simple linear fundamental diagram:

$$Q(\rho) = \begin{cases} C_{free} \cdot \frac{\rho}{\rho_{crit}} & \rho \leq \rho_{crit} \\ C_{queue} \cdot \frac{\rho_{jam} - \rho}{\rho_{jam} - \rho_{crit}} & \rho > \rho_{crit} \end{cases} \tag{11}$$

In Eq. (11), C_{free} denotes the free flow capacity (typically 2250 veh/h/lane), while C_{queue} denotes the queue discharge rate (between 1800 and 2000 veh/h/lane); ρ_{crit} denotes the critical density (25 veh/km/lane), and finally ρ_{jam} denotes the jam density. Note that for $\rho \leq \rho_{crit}$ we have $c(\rho) = C_{free}/\rho_{crit}$. For $\rho > \rho_{crit}$ we have $c(\rho) = C_{queue}\rho_{jam}\ (\rho_{jam} - \rho_{crit})$. This shows clearly how the parameters of the fundamental diagram determine the way perturbations propagate through the flow.

3.2 Specification of the Transition Probability Rate

We will use the following linear expression for the rate $\pi(\rho, P)$ (both for the F-S transitions, and the S-J transitions, be it with different parameter values):

$$\pi(\rho, P) = \begin{cases} (\pi_0 + \pi_1 P) \cdot \frac{\rho - \rho_0}{\rho_1 - \rho_0} & \text{for } \rho_0 \leq \rho \leq \rho_1 \\ 0 & \text{elsewhere} \end{cases} \tag{12}$$

Additionally, we will assume that $P = 0$, if the density is less than ρ_0. Furthermore, P will be limited to values between 0 and 1. After some straightforward computations, it follows that along the characteristic curves $C = \{t(s), x(s)\}$, the transition probability equals:

$$P(s) = \min\left\{ \frac{\pi_0}{\pi_1} \left(e^{\pi_1 \left(\frac{\rho - \rho_0}{\rho_1 - \rho_0} \right) s} - 1 \right), 1 \right\} \tag{13}$$

3.3 F-S Transition Probability Behavior

Let us consider two-lane 10 km road with an on-ramp at $x = 6$ km. For the piecewise linear fundamental diagram, we assume $C_{free} = 4500$ veh/h, $C_{queue} = 4000$ veh/h, $\rho_{crit} = 50$ veh/km and $\rho_{jam} = 250$ veh/km. For the scenario at hand, we choose $Q_{main} = 3500$ veh/h and $Q_{on-ramp} = 1250$ veh/h. After playing around with different parameter values, we have chosen $\pi_0 = 1$ and $\pi_1 = 100$ (for illustration purposes); $\rho_0 = 40$ veh/km and $\rho_1 = \rho_{crit}$.

Fig. 1 below shows the results of the numerical experiment. It shows the density profile, the location of the points on the fundamental diagram, and the transition probabilities. The F-S breakdown probability increases non-linearly after the on-ramp at $x = 6$ km. In other words, the occurrence of a breakdown becomes more likely further downstream of the on-ramp.

If we, for the sake of argument, assume that we would have an F-S transition (in this case, modeled by temporarily assume that the capacity is reduced from C_{free} to C_{queue}) when $P_{F-S} > 0.5$, the simulation shows that at a certain time instant, the transition occurs (downstream of the bottleneck), moves upstream, and passing the on-ramp location. There it leads to the on-set of congestion (because the capacity is reduced); see Fig. 2.

3.4 S-J Transition Probability Behaviour

For the S-J transitions, we find similar behavior. In this case, we have again used $\pi_0 = 1$ and $\pi_1 = 100$ (for illustration purposes) to describe the S-J transition; $\rho_0 = \rho_{crit}$ and $\rho_1 = 200$ veh/km. For this scenario, we assume that $Q_{main} = 4000$ veh/h and $Q_{on-ramp} = 1500$ veh/h, implying that the bottleneck is oversaturated.

The result is indeed similar to the result we found for the F-S transition (see Fig. 3): the probability on an S-J transition is zero at the on-ramp (where

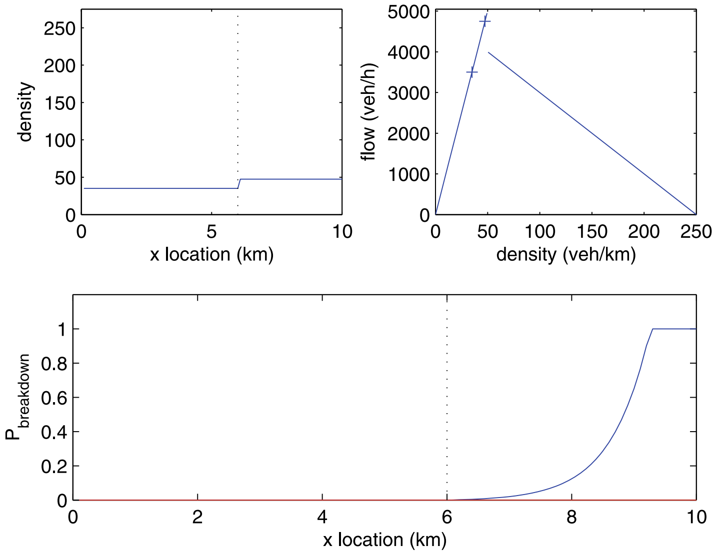

Fig. 1. On-ramp scenario showing increase in the F-S transition probability.

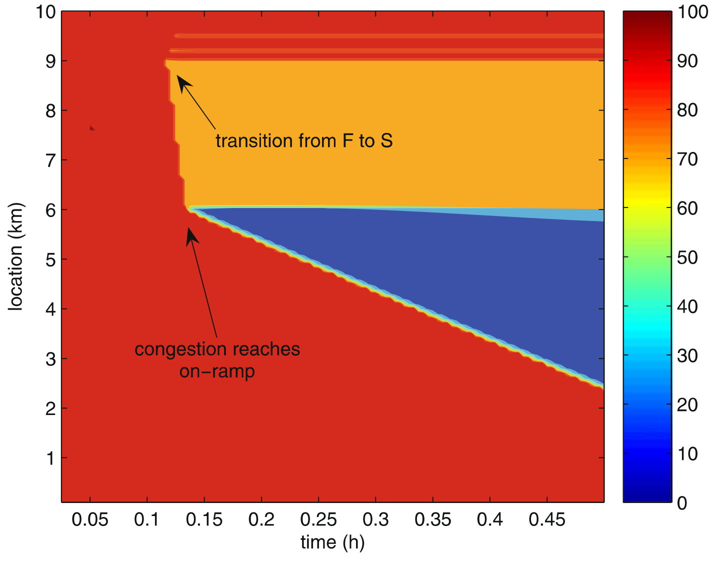

Fig. 2. Speed contours showing F-S transition. Figure shows how congestion sets in downstream of on-ramp and moves upstream.

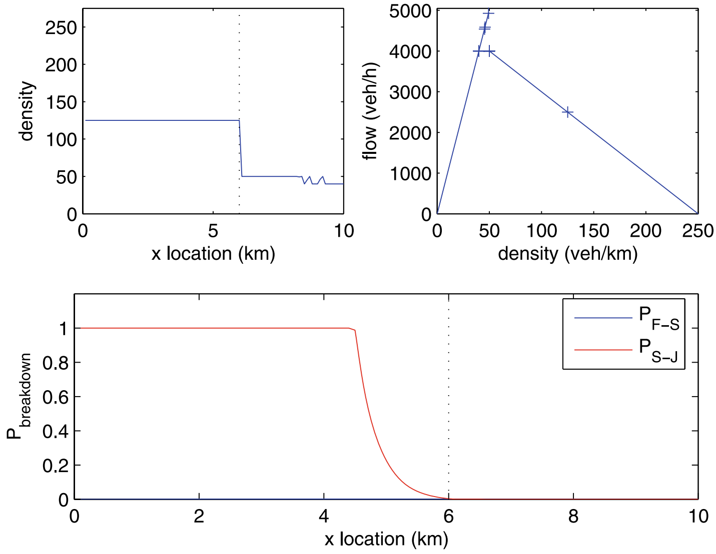

Fig. 3. Probability of an S-J breakdown, increasing as we move from the on-ramp in the upstream direction.

this model assumes that the head of the queue is located), and increases non-linearly as we move upstream away from the bottleneck.

As a final example, let us assume that an S-J transition occurs when $P_{S-J} > 0.5$ (i.e. it is in a way deterministic). Fig. 4 shows the results of this analysis. Clearly, the precise values are not realistic, but the general picture appears to be correct. It is also interesting to note the chaotic-like patterns that emerge even when this simple example is used.

4 Conclusions and Future Work

In this paper we have proposed a relatively simple extension of the first-order model pertaining to the inclusion of breakdown probabilities. The breakdown probability is modeled using a partial differential equation. The main assumption is that information regarding the breakdown probability moves along the characteristic curves.

The workings of the model are illustrated by means of an example featuring flow breakdown due to an on-ramp. Using this example, it is shown that the model can capture the main features of the different phase transitions (free flow to synchronized flow, synchronized flow to jam).

Future research is aimed at modeling the phase-transition itself. In the examples provided in this paper, this was achieved using a simple threshold value for the transition probabilities. A stochastic approach is however more realistic. Clearly, this would yield a stochastic first-order macroscopic model.

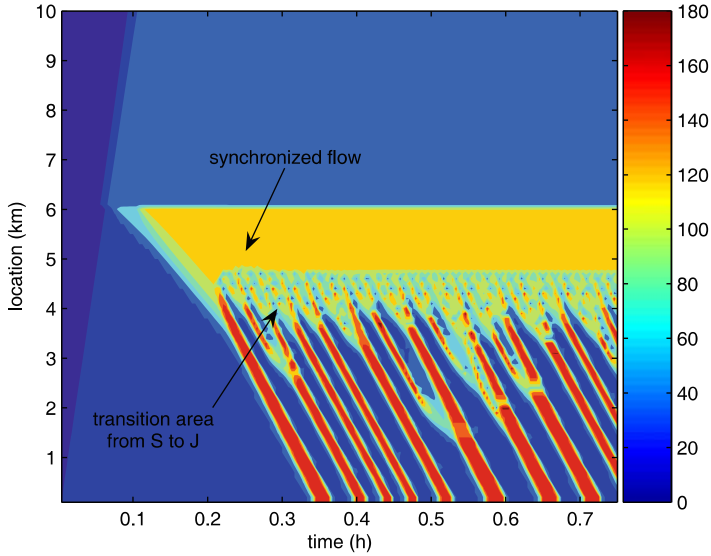

Fig. 4. Speed contours showing transition from S to J upstream of the on-ramp.

Another extension of the theory is to use a multi-class traffic flow model, distinguishing between person-cars and trucks. In doing so, the dynamics of the phase-transitions can be made dependent on the traffic composition, since clearly this has a strong effect on the breakdown probability dynamics.

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