A Kinematic Wave Model to Reproduce Capacity Drop

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***Abstract*—The science of traffic jams indicates that traffic operations can be a promising direction to dissolve jams. The finding of capacity drop phenomena theoretically provides a fundamental basis for efficient freeway control strategies. The capacity drop means that the queue discharge rate is lower than the capacity on freeways. New empirical findings show that the queue discharge rate increases as the speed in congestion increases. Those phenomena has been widely observed not only at freeway merge bottlenecks but also on homogeneous road sections. So a good kinematic wave model should be required to firstly reproduce different queue discharge rates depending on congestion states, and secondly capture the capacity drop on both homogeneous sections and merges on freeways. However, no model has made it until now. Therefore, this paper proposes a new kinematic wave model to fill in the gap. The new model can reproduces different queue discharge rates corresponding to the vehicular speed in congestion in the same simulation. By utilizing the Lagrangian formulation and the upwind discretization method, this paper presents a link model which captures the capacity drop of stop-and-go waves. This model can also be extended to reproduce capacity drop at freeway merge bottlenecks. This model can be applied to develop freeway operations, e.g. speed limit strategies for eliminating stop-and-go waves.**

I. INTRODUCTION

As the number of vehicles grows, a majority of cities over the whole word are suffering serious traffic congestion. Traffic jams have been a daily phenomenon on freeways around urbanized zones, which cause considerable lost of travel time and serious pollution. Researchers have devoted much effort to solving the congestion problem. One of the most promising solutions is the traffic management. Empirical studies provide theoretical bases for promising freeway operations. One of those most crucial empirical findings is the capacity drop phenomenon.

The capacity drop means that on a freeway the queue discharge rate is frequently lower than the capacity. The queue discharge rate, which is referred to as effective capacity in some literatures, is the maximum possible flow in the downstream of jams. The capacity is the maximum possible flow. This paper defines the 'flow' as the number of vehicles passing through one location per unit of time, which is referred to as 'flow rate' or 'volume' in other papers. We divide congestion into two categories: stop-and-go waves which propagate upstream with both two congestion fronts and standing queues whose heads are fixed at bottlenecks. The capacity drop of standing queues at on-ramp bottlenecks indicates benefits from on-ramp metering. The capacity drop of stop-and-go waves can contribute to efficient speed limit strategies [1]. Therefore, to reproduce the capacity drop of congestion including both of standing queues and stop-and-go waves is quite important for traffic control designs.

This paper wants to highlight that reproducing capacity drop not only should capture a qualitative difference between the capacity and the queue discharge rate, but also should indicate quantitatively to what extent the queue discharge rate will decrease once a queue forms in the upstream. Because the accuracy of the reproduced capacity drop can influence the values of control variables and even the strategy performances. Since Yuan et al. [2] have revealed a relation between the speed in congestion and the queue discharge rate, so the accuracy of the model can be improved by capturing this relation when reproducing the capacity drop.

To date, most of researchers apply second order model to reproduce capacity drop in simulations, such as in [3], [4], [5]. Researchers have realized and highlighted the high accuracy of second order model [6]. However, the improved accuracy of the second order model is at an expense of complexity. The increased complexity should not be ignored in traffic control designs. By contrast, a first order model is very simple which only employs a fundamental diagram and a vehicle conservation law. Therefore, some researchers have tried to model the capacity drop with a kinematic wave model in an assumption of discontinuous fundamental diagram, such as a reverse-lambda fundamental diagram. But as argued in [7], it is best to utilize a continuous fundamental diagram rather than a discontinuous one.

Some researchers have tried to incorporate capacity drop into a first order model with continuous fundamental diagram. Roncoli et al. [8] reproduce the capacity drop phenomenon on on-ramp bottlenecks. But this model can be applied for the optimization of on-ramp metering, but can not be applied on situations without merging. Srivastava et al. [9] utilize a fundamental diagram with two values of capacity. This one reproduces the capacity drop by choosing different capacities. Alvarez-Icaza and Islas [10] apply a hysteresis cycle to reproduce the capacity drop. However, both of models in [9]

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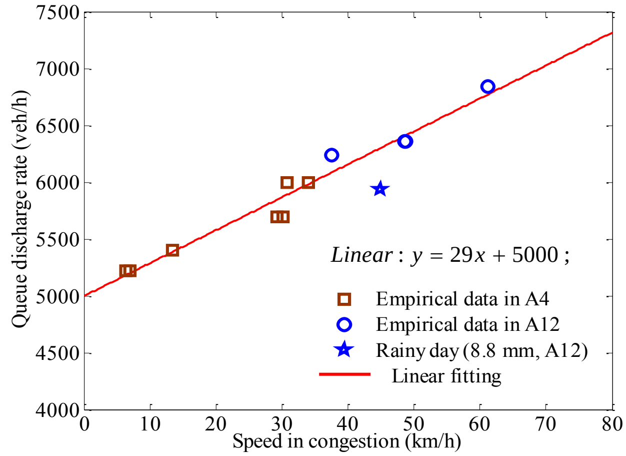


Fig. 1. Empirical data showing a relation between the speed in congestion and the queue discharge rate [2]

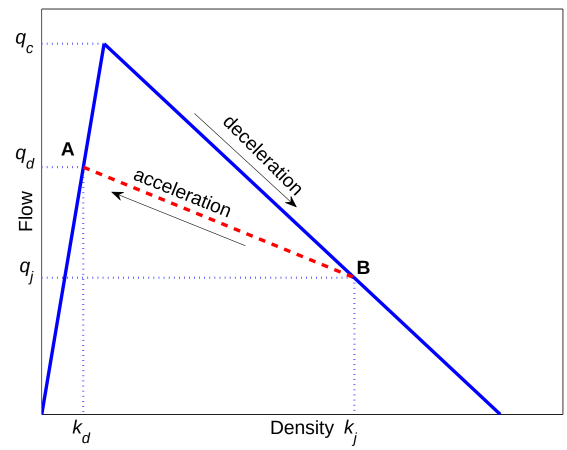


Fig. 2. Fundamental diagram with capacity drop

and [10] can not reproduce the relation between the queue discharge rate and the congestion states, e.i., the capacity drop magnitude is fixed, independent from the congestion states. In other words, the accuracy of all above models is not high enough for control designs. As far as authors know, until now existing first order models firstly can not reproduce a wide range of capacity drop in one simulation, and that secondly many are limited to on-ramp bottlenecks because they are oriented towards on-ramp metering optimizations. Therefore, this paper fills in the gap by presenting a kinematic wave model which can reproduce a wide-range capacity drop magnitudes in one simulation. The reproduced capacity drop depends on congestion states. If we simulate two different stop-and-go waves on one freeway at the same time, then the queue discharge rates differ. So is the shock wave speed. This model can be extended to capture capacity drop at freeway merge bottlenecks.

The model in this paper applies hysteresis circle, which is introduced in [11], to capture capacity drop. The congestion branch in the fundamental diagram is divided into acceleration and deceleration branches. This model assumes a clockwise hysteresis circle to describe the process of passing through a jam. The deceleration branch is fixed while the shock wave speed of the acceleration branch is a function of the congestion states. So the congestion state determines the queue discharge rate which corresponds to recent empirical findings. To solve this kinematic wave model, we express this model in the Lagrangian coordinate and discretized it with upwind discretization method.

We organize this paper in four sections. Section II describes the new kinematic wave model for capacity drop and its solutions in Lagrangian coordinate. Section III presents the simulation setting and simulation results. Finally, Section IV draws the conclusions of this paper.

II. Model Formulation

A. Kinematic Wave Model to reproduce capacity drop

The first-order kinematic wave model should follow the conservation law of vehicles. Let $q = q(x,t)$, $v = v(x,t)$ and $k = k(x,t)$ denote the traffic flow, speed and density respectively at time instant t and location x. The vehicle conservation law means no vehicles are created or lost. On a homogeneous freeway the law can be presented as (1).

$$\frac{\partial k(x,t)}{\partial T} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{1}$$

T is the simulation time, $T = t \cdot \Delta T$. ΔT is the temporal length of each time instant. In equilibrium states, traffic flow, density and speed follow:

$$v(x,t) = \frac{q(x,t)}{k(x,t)} \tag{2}$$

To integrate capacity drop into the kinematic wave model, Let us assume a triangular fundamental diagram and a clockwise hysteresis circle, see Fig. 2. The hysteresis circle divides the congestion branch into an acceleration and deceleration branch. The acceleration branch is below the deceleration one. When vehicles accelerate from congestion, the fundamental relation between flow and density will follow the acceleration branch which is shown as a red dash line in Fig. 2. In other words, when vehicles' speed remain the same or decrease, the fundamental relation will follow the traditional triangular fundamental diagram which is shown as blue bold line. This assumption satisfies the requirement of a continuous fundamental diagram. This requirement has been discussed in [7]. Let Q denotes the relation between q and k in fundamental diagram when vehicles are not accelerating, $q = Q(k)$. Otherwise the fundamental relation will be superscripted by prime, $q = Q^{'}(k)$. For simplicity, we refer the traditional triangular fundamental diagram to as deceleration branch in this paper.

In Fig. 2, there are two intersections where the acceleration branch meets the deceleration branch, state A and B. State A indicates that the downstream free flow of one congestion.

State B indicates the congestion where vehicle leave for state A. The queue discharge rate q_{d}, corresponding to state A, is smaller than the capacity, q_{c}, which presents the capacity drop. Meanwhile, these two intersections present the continuity of the fundamental diagram.

The slope of the deceleration branch is fixed while that of the acceleration branch is not. The slope of the acceleration branch depends on the congestion states, which follows a linear relation between the speed in congestion v_{j} and the queue discharge rate q_{d} (see Fig. 1), as suggested by [2]. Following this linear relation, this model applies two parameters, a and b, to show the relation between the speed in congestion and the queue discharge rate, see (3). Note that because free-flow speed in triangular fundamental diagram is a single value rather than a wide range of values, so free-flow speed which is slightly lower than the v_{f} will be treated as the speed in congestion. Then a queue discharge rate which is higher than the capacity will be calculated, which does not exist in reality. Hence, the queue discharge rate in our model is calculated as the minimum value between the capacity and the calculated queue discharge rate. Finally, once knowing the traffic jam state B, this model can deduce the downstream free-flow state A, i.e., the queue discharge rate $q_{\mathrm{d}}(k_{\mathrm{j}})$. We can get:

$$q_{\mathrm{d}}(k_{\mathrm{j}}) = \min\left(C, a \cdot v_{\mathrm{j}}(k_{\mathrm{j}}) + b\right) \quad (3)$$

k_{j}, $q_{\mathrm{j}}(k_{\mathrm{j}})$ and $v_{\mathrm{j}}(k_{\mathrm{j}})$ is the density, flow and vehicle speed in the congestion state B, respectively. v_{f} is the free-flow speed. Since state A lies in the free-flow branch, the density $k_{\mathrm{d}}(k_{\mathrm{d}})$ in state A is:

$$k_{\mathrm{d}}(k_{\mathrm{d}}) = \frac{q_{\mathrm{d}}(k_{\mathrm{j}})}{v_{\mathrm{f}}} \quad (4)$$

Equation (3) and (4) indicate that we can deduce state $A(q_{\mathrm{d}}(k_{\mathrm{j}}), k_{\mathrm{d}}(k_{\mathrm{j}}))$ according to state $B(q_{\mathrm{j}}(k_{\mathrm{j}}), k_{\mathrm{j}})$. All parameters in state A depend on those in state B. Let $\omega^{'}$ stands for the shock wave speed of the acceleration branch:

$$\omega^{'}(k_{\mathrm{d}}) = \frac{q_{\mathrm{d}}(k_{\mathrm{j}}) - q_{\mathrm{j}}(k_{\mathrm{j}})}{k_{\mathrm{d}}(k_{\mathrm{j}}) - k_{\mathrm{j}}} \quad (5)$$

$\omega^{'}(k_{\mathrm{d}})$ is a variable, as a function of k_{j}. Equation (5) indicates that with different congestion states $B(q_{\mathrm{j}}, k_{\mathrm{j}})$, we can deduce different corresponding free flow states $A(q_{\mathrm{d}}(k_{\mathrm{j}}), k_{\mathrm{d}}(k_{\mathrm{j}}))$ through different acceleration branches. That is, in this model, both of the maximum jam density and the deceleration branch are fixed while the acceleration branch changes depending on congestion states. Fig. 2 and Fig. 3 only show one acceleration branch as an example.

This paper reformulates the new kinematic wave model in the Lagrangian formulation. Because firstly the Lagrangian coordinate can lead to a more efficient numerical method than Eulerian. Secondly we apply upwind discretization method to solve the kinematic wave model in Section II-B. For the full description of applying the Lagrangian coordinate to the kinematic wave model we refer to the literature [12].

In Lagrangian coordinate system, the Eulerian conservation equation (1) is reformulated as (6):

$$\frac{\partial s(x,t)}{\partial T} + \frac{\partial v(x,t)}{\partial N} = 0 \quad (6)$$

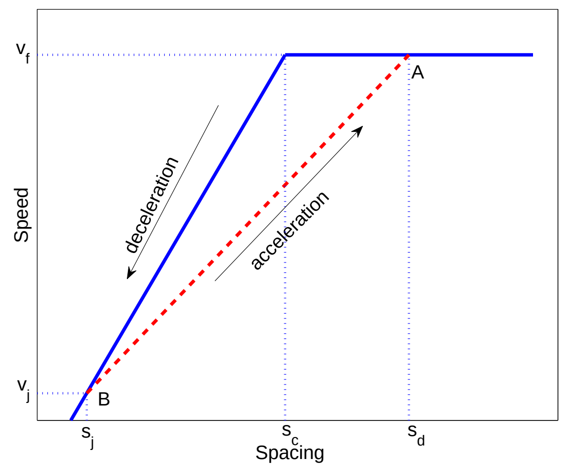


Fig. 3. Fundamental diagram with capacity drop in Lagragian coordinate

N is the cumulative number of vehicles, which decreases in space. ΔN is the number of vehicles in each Lagrangian cell. In this paper, $\Delta N > 0$. v is the vehicular speed that is a function of spacing s:

$$v = \begin{cases} V^{'}(s), & \text{for acceleration} \quad (7) \\ V(s), & \text{for the others} \quad (8) \end{cases}$$

which is the fundamental diagram in the Lagrangian coordinate, see Fig. 3. The prime indicates the acceleration branch. The blue bold line is the deceleration branch while the red dash line is the acceleration branch. The maximum speed v_{f} is the free-flow speed. The slop of the deceleration branch is

$$\max\left|\frac{\partial v}{\partial s}\right| = w k_{\max} \quad (9)$$

$k_{\max}$ is the jam density in the most heavy congestion state. The minimum spacing is the reverse of the maximum jam density $\frac{1}{k_{\max}}$. The deceleration branch is fixed while the acceleration branch varies depending on the congestion state B, too. The queue discharge rate is formulated into the vehicular spacing downstream of the congestion. The lower the queue discharge rate, the larger the spacing. The coordinates of state A in the Lagrangian coordinate are $(s_{\mathrm{d}}, v_{\mathrm{f}})$ and

$$s_{\mathrm{d}} = \frac{v_{\mathrm{f}}}{a \cdot v_{\mathrm{j}} + b} \quad (10)$$

v_{j} is the vehicular speed in congestion and s_{d} is the vehicular spacing in the downstream free flow. The slope of the acceleration branch in the fundamental diagram and s_{d} fully depends on v_{j}.

Only five parameters in total in the new kinematic wave model need calibrations, that is, free flow speed, capacity, shock wave speed, a and b. The additional parameters, except normal parameters of triangular fundamental diagram, are only a and b. a indicates by how many vehicles per hour the traffic jam discharge rate will increase when the speed in the jam grows by 1 km/h. b is the queue discharge rate of wide moving jam in which vehicles' speed equal to zero km/h.

To calibrate the a and b, we assume that a and b are independent from the fundamental diagram. So we can calibrate a

and b independently. Based on a hypothesis of a linear function shown in Fig. 1 which is from [2], the calibration of a and b only needs two traffic jams and their discharge rates. The more jams, the more accurate the calibration. The approach to collect and analyze empirical data to get relation between speed in congestion and queue discharge rate is described in details in [2] and [13].

B. Solution to the kinematic wave model

To solve the kinematic wave model, this paper applies an upwind discretization method in Lagrangian coordinate. This method has been described fully in literatures [14] and [15]. This section firstly describe the upwind method briefly. Then we extend this method for solving the new kinematic wave model.

1) Basic upwind method: The upwind method divides vehicles and time into vehicular groups and time steps respectively. Each vehicular group is referred to as a Lagrangian cell. Each cell i is characterized by three quantities:

- number of vehicles ΔN
- cell spacing $s(t)$
- speed $v(t)$

For stability and convergence, the time length should follow CFL condition (11):

$$\frac{\Delta N}{\Delta T} \geq \max \left| \frac{\partial v}{\partial s} \right| \tag{11}$$

When the CFL condition is satisfied as an equality, the upwind method can be free of numerical errors.

Subscripts are used to indicate the cumulative number of Lagrangian cells. Cell i follows cell $i-1$. The upwind method is expressed in time discretization as:

$$s_i(t+1) = s_i(t) + (v_{i-1}(t) - v_i(t)) \cdot \frac{\Delta T}{\Delta N} \tag{12}$$

The numerical scheme (12) is equivalent to the following expression:

$$x_i(t+1) = x_i(t) + v_i(t) \cdot \Delta T \tag{13}$$

which updates the new position x of the cell i.

2) Extended solution: This section describes the extension of the basic upwind discretized method for solving the new kinematic wave model.

The extension is the new process of updating each cell speed. The cell speed updating process is presented in Fig. 4. V_i^* is the default fundamental speed-spacing relation, following which cell i updates its speed at present. V_i^* get updated in every time step, too. This process firstly update the cell's position and spacing. Then, it checks which speed-spacing branch should the cell follows. This checking process is highlighted with a dashed rectangle in Fig. 4. If $v_\mathrm{i}(t) = v_\mathrm{f}$, the speed updating process should follow the deceleration branch $V_\mathrm{i}^* = V$. If not, firstly the process is going to update the speed with the default speed-spacing relation which has been updated in the last time step. Then secondly the process checks whether an acceleration branch should be or have been

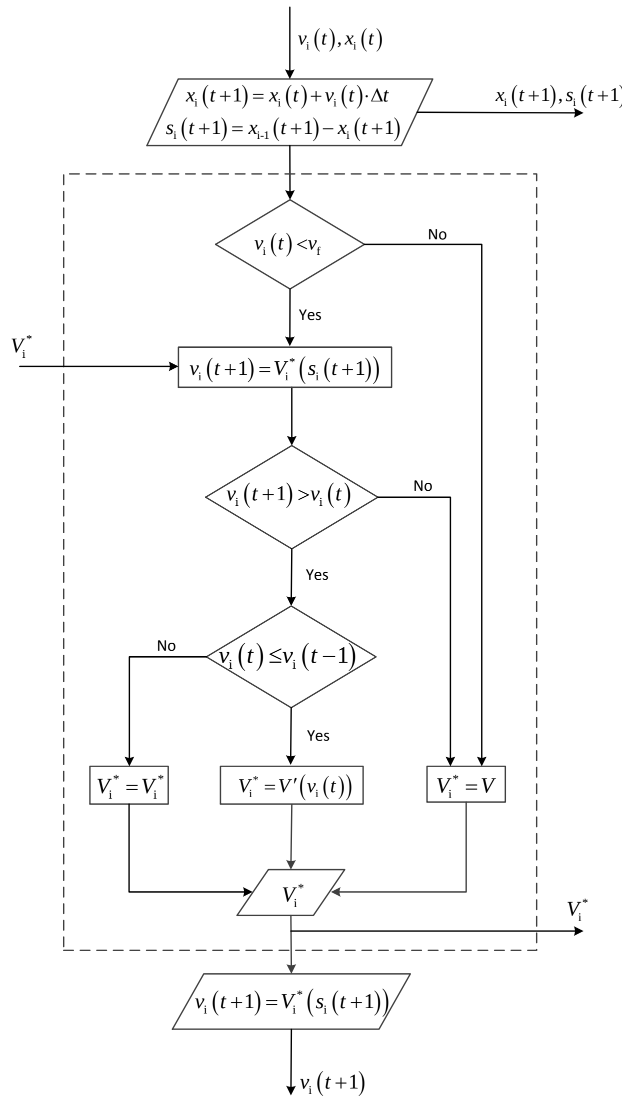


Fig. 4. The process of updating cell speed

activated. If the acceleration branch has been activated, then the default relation V_i^* remains the same as that in the last time step. If this branch should be but has not been activated yet, then it means $v_\mathrm{i}(t)$ is the speed in congestion. The default relation will be updated into an acceleration branch. Finally the process updates the cell speed again following the newly updated speed-spacing relation V_i^*.

When applying the upwind discretization method in the Lagrangian coordinate to solve the kinematic wave model, numerical errors inevitably arises. Without capacity drop, the upwind method can be free of numerical by setting an equal sign in CFL condition, see(14).

$$\Delta T = \Delta N \cdot \max \left| \frac{\partial s}{\partial V(s)} \right| \tag{14}$$

However, because the slop of the acceleration branch $V^{'}(s)$ is smaller than that of the deceleration branch $V(s)$, see Fig. 3, so:

$$\Delta T < \Delta N \cdot \max \left| \frac{\partial s}{\partial V'(s)} \right| \quad (15)$$

No equal sign in (15) means that when vehicles accelerate from congestion, numerical errors will be inevitable.

Numerical errors are much worse when applying minimum supply demand method than upwind method. Therefore, this paper only applies upwind discretization method as the solution. The low accuracy of minimum supply demand method should not be a surprise. For the comparison between accuracy of both discretization methods, we refer to literature [15].

Since we have noticed that the minimum supply demand method (or the Cell Transmission Model [16], [17]) is still one of the most widely used methods (or models), this paper wants to highlight that fortunately our model can still be solved with minimum supply demand method or incorporated with Cell Transmission Model easily by revising the demand and supply function.

The solution to this model can be extended to model the capacity drop at a freeway merge bottleneck. Because as shown in [18], each merging vehicle is a moving bottleneck which emerges a stop-and-go wave at the merge bottleneck. The standing queue at a merge bottleneck can be seen as several stop-and-go waves. So it is highly possible to model the capacity drop at a merge bottleneck once we have been able to model that of a stop-and-go wave. The extension is being studied.

III. Simulation

For the fundamental diagram, this paper lets $v_{\mathrm{f}} = 114$ km/h, $q_{\mathrm{c}} = 6840$ veh/h, $k_{\mathrm{c}} = 60$ veh/km and $\omega = 18$ km/h. Those are typical values for an empirical fundamental diagram on a three-lane freeway in the Netherlands. Accordingly the critical spacing $s_{\mathrm{c}} = \frac{1}{60}$ km, see Fig. 3. For simplicity we set $\Delta N = 1$, that means our solution to the new kinematic wave model is a car-following model. Please note that we can solve the kinematic wave model in the same way by setting $\Delta N > 1$. The microscopic model is just an example of the solutions.

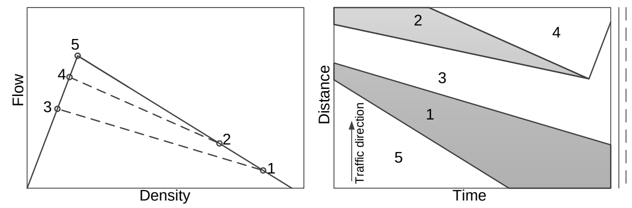
Numerical errors can be minimized when by setting $\Delta t = \frac{1}{\omega k_{\mathrm{j}}} = \max \left| \frac{\Delta s}{\Delta v} \right|$ to satisfy CFL condition (11) as an equality. Parameters $a = 29$ and $b = 5000$ in (3), which are the same as suggested in [2].

In our simulations, two jams are considered:

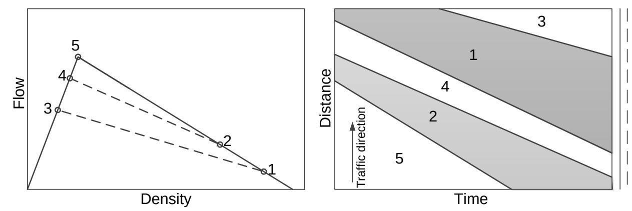
- Both of congestions are stop-and-go waves J_{s} and J_{f} which propagate on a homogeneous freeway
- Heavy congestion J_{s}: density $k_{\mathrm{J}}^{s} = 400$ veh/km
- Light congestion J_{f}: density $k_{\mathrm{J}}^{f} = 200$ veh/km
- Subscript s indicates slow vehicle while f indicates fast vehicle

Two typical cases representing that those two jams propagating on homogeneous freeway is simulated.

- Case 1: J_{s} propagates in the upstream of J_{f}



(a) Case 1: Low-speed jam J_{s} upstream



(b) Case 2: Low-speed jam J_{s} downstream

Fig. 5. Shock wave analysis in a) Case 1 and b) Case 2. In each case, the left figure shows the fundamental diagram and the right one the speed contour

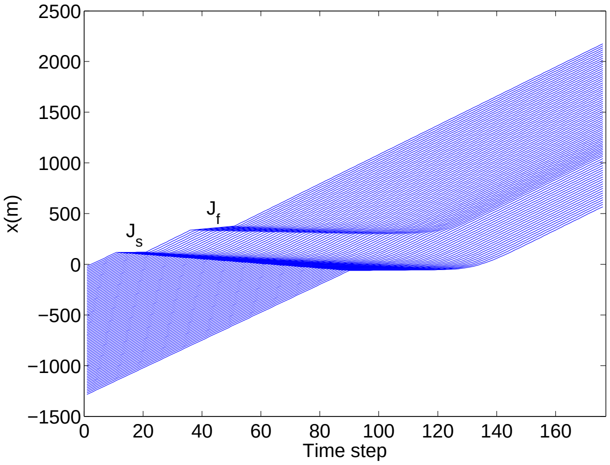
- Case 2: J_{f} propagates in the upstream of J_{s}

As shown in Section II, the queue discharge rate in state A is determined by the congestion state B. So the outflow of jams J_{s} and J_{f} should be different. Fig. 5 shows shock wave analyses on both of these two cases. State 1 and 2 indicate the stop-and-go wave J_{s} and J_{f}, respectively. Density (color) in J_{s} is higher (darker) than that in J_{f}. State 3 and 4 are the downstream free-flow states of jam 1 and 2, respectively. State 5 indicates the capacity. The shock wave analysis predicts the difference between those two case simulations. This difference is not a surprise because the queue discharge rate of the upstream jam will be the inflow of the downstream jam. Those two case studies can test that whether our new model can capture the capacity drop and its fundamental features.

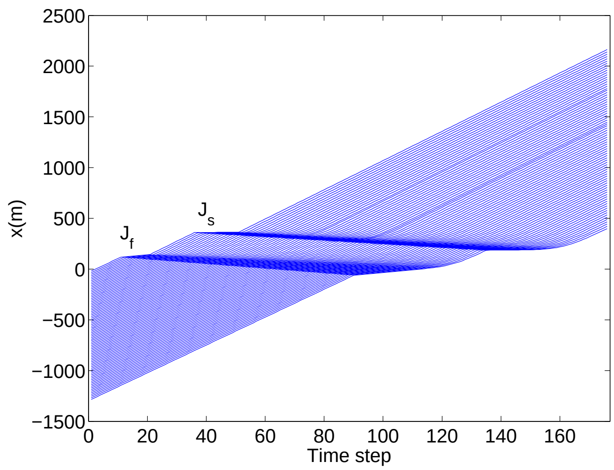
To emerge those two stop-and-go waves, we simply set the time series of vehicles' spacing and speed of the first Lagrangian cell. At the beginning of the simulation ($t = 0$), all the cell spacings are set to be critical spacing and all the cell speed are free-flow speed.

Simulation results in both of study cases are presented in Fig. 6. The results are exactly the same as shown in Fig. 5. The blue lines are the Lagrangian cells. Because in this simulation we set $\Delta N = 1$, so they are vehicles' trajectories, too. The queue discharge rate of J_{s} is 5052 veh/h (capacity drop = 26%) and that of J_{f} is 5626 veh/h (capacity drop = 18%). In Case 1, the width of the stop-and-go wave J_{s} increases while the stop-and-go wave J_{f} is diminished. For J_{s}, due to capacity drop the inflow (equals to capacity) is higher than the outflow, therefore both of the queue length and time delay increase. For J_{f}, because the queue discharge flow of J_{s} is lower than the inflow which is the outflow of J_{f}, so finally J_{f} dissipates until it disappears. For the same reason, in Case 2 both of queues grow. Simulations show our model works well when capture the capacity drop and its features.

As predicted in Section II-B, Fig. 6 presents no numerical error in the upstream front of J_{s} and J_{f} in Case 1 and Case



(a) Case 1: Low-speed jam J_s upstream



(b) Case 2: Low-speed jam J_s downstream

Fig. 6. Simulation results in a) Case 1 and b) Case 2

2, respectively. In the further downstream, numerical problems arise as predicted in Section II-B.

IV. Conclusion

This paper presents a new kinematic wave model to reproduce the capacity drop phenomenon. This new model uses a clockwise hysteresis circle to capture the capacity drop. It shows advantages over other first order models because firstly it can reproduce the capacity drop and its significant features at the same time, such as the relation between the speed in congestion and the queue discharge rate. Secondly, the reproduction of the capacity drop can be applied for stop-and-go waves, which makes it can be extended to the standing queues. To minimize the numerical errors, we reform the kinematic wave model in Lagrangian coordinate and solve it with the upwind discretization method. Though the solution to the new kinematic wave model in this paper is still only a link model, it is possible to extend the solution to node models. In the future authors are going to extend the solution to make it.

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